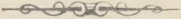


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N^o 36



THE ELEMENTS OF
PLANE TRIGONOMETRY



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THE ELEMENTS OF PLANE TRIGONOMETRY

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PREFACE

OUR experience of teaching trigonometry in schools has convinced us that no difficulty arises from introducing, at the very beginning of the course, the circular functions of angles of any magnitude, and that concentration on the acute angle leads to unnecessary difficulties, and sometimes to erroneous ideas of the circular functions. We are confident that progress is more rapid by adopting the plan of this book.

Our chief desire has been to produce a teaching book. The space occupied by bookwork has been kept as small as possible, while the guidance afforded by notes and by worked examples should be sufficient even for the private student.

The grading of examples has received much attention, and, while the majority of the examples meets the needs of the average student, there are many examples to test the more advanced.

The sequence of the book is, in our opinion, the natural teaching order. The chapter on equations is toward the end of the book, but it is divided into sections, so that the different types of equations may be taken as soon as the necessary formulæ have been learned.

Graphical work has received generous treatment, with emphasis on the graphical solution of equations.

The book is intended to cover the work done in secondary schools in preparation for examinations of the standard of the Oxford and Cambridge Local, the Scottish Leaving Certificate, etc., and to meet the requirements of junior classes in universities and technical colleges.

To Professor D. B. M'Quistan we express our gratitude for kindly reading most of the work in manuscript and for

many valuable suggestions. We are also indebted to Mr G. Sommerfeldt, B.Sc., of Allan Glen's School, for assistance in reading the proof sheets.

We shall be grateful to receive intimation of any errors which may be found in the book, or any suggestions for improvement.

J. T. B.

A. M.

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CHAPTER I

MEASUREMENT OF ANGLES

§ 1. SEXAGESIMAL MEASURE

The geometrical unit of angle is the right angle, which is defined thus : when one straight line stands on another straight line so that the adjacent angles are equal, each of these adjacent angles is a right angle. The right angle has been subdivided as indicated in the following table :

$$\begin{aligned} 60 \text{ seconds} &= 1 \text{ minute} \\ 60 \text{ minutes} &= 1 \text{ degree} \\ 90 \text{ degrees} &= 1 \text{ right angle} \end{aligned}$$

An angle of 79 degrees 43 minutes 57 seconds is written $79^{\circ} 43' 57''$, and an angle expressed in these units is said to be expressed in sexagesimal measure.

Note. The right angle has been subdivided into 100 grades, the grade into 100 minutes, and the minute into 100 seconds. This is the centesimal method, but it has never been in common use.

Example 1. Express in sexagesimal measure 0.032706 right angle.

$$\begin{array}{r} 0.032706 \text{ right angle} \\ \hline 90 \\ \hline 2.94354 \text{ degrees} \\ \hline 60 \\ \hline 56.6124 \text{ minutes} \\ \hline 60 \\ \hline 36.744 \text{ seconds} \end{array}$$

$\therefore 0.032706$ right angle $= 2^{\circ} 56' 37''$ (to nearest second).

Example 2. Express as the decimal of a right angle $43^{\circ} 54' 29''$.

$$\begin{aligned} \frac{43^{\circ} 54' 29''}{90^{\circ}} &= \frac{2634' 29''}{90 \times 60 \times 60''} \\ &= \frac{158069''}{90 \times 60 \times 60''} \\ &= 0.487867 \text{ (to six decimal places).} \end{aligned}$$

EXERCISES

1. Express the following angles in sexagesimal measure :

- (a) 0.543086 right angle.
- (b) 3.217835 right angles.
- (c) 1.069243 right angles.

2. Express the following angles as a decimal of a right angle :

- (a) $79^\circ 43' 57''$.
- (b) $35^\circ 11' 45''$.
- (c) $142^\circ 35' 16''$.

§ 2. CIRCULAR MEASURE

*Theorem I.*¹ The ratio of the length of the circumference to the length of the diameter is constant for all circles.

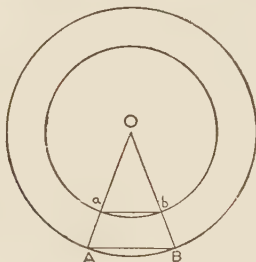


FIG. 1

Place any two circles so that they have a common centre O (Fig. 1). Let OA and OB , two radii of the outer circle, contain angle $\frac{360^\circ}{n}$, where n is any integer, and let OA and OB cut the inner circle at a and b respectively. Join AB , ab .

Then, since $\hat{O} = \frac{360^\circ}{n}$, AB and ab are sides of regular n -gons inscribed in the outer and inner circles respectively.

But Δ 's OAB , Oab , have $\hat{O}$ common, and $\frac{OA}{OB} = \frac{Oa}{Ob}$ and are therefore similar.

$$\therefore \frac{OA}{Oa} = \frac{AB}{ab}$$

¹ In a first course the student may omit the proof of Theorem I.

$$= \frac{n \cdot AB}{n \cdot ab}$$

$$= \frac{\text{perimeter of regular } n\text{-gon inscribed in outer circle}}{\text{perimeter of regular } n\text{-gon inscribed in inner circle}}$$

If n increases indefinitely the n -gons tend to become the circles.

$$\therefore \frac{OA}{Oa} = \frac{\text{circumference of outer circle}}{\text{circumference of inner circle}},$$

$$\therefore \frac{\text{circumference of outer circle}}{OA} = \frac{\text{circumference of inner circle}}{Oa},$$

$$\therefore \frac{\text{circumference of outer circle}}{\text{diameter of outer circle}} = \frac{\text{circumference of inner circle}}{\text{diameter of inner circle}},$$

$$\therefore \frac{\text{circumference}}{\text{diameter}} \text{ is constant for all circles.}$$

Note 1. The constant ratio proved above is denoted by the symbol π . Its value to five places of decimals is 3.14159, but when great accuracy is not required its value may be taken as $\frac{22}{7}$.

Note 2. Using the symbol π , Theorem I can be stated briefly thus:

$$\text{circumference} = \pi \times \text{diameter},$$

$$\text{or circumference} = 2\pi \times \text{radius}.$$

DEFINITION. The angle subtended at the centre of a circle by an arc equal in length to the radius is called a radian.

Theorem II. The radian is a constant angle.

Let arc AB = radius OA (Fig. 2).

$\therefore \angle AOB$ is a radian.

$$\text{But } \frac{\angle AOB}{180^\circ}$$

$$= \frac{\text{arc } AB}{\text{semi-circumference}} \quad (\text{angles at centre are proportional to arcs subtending them})$$

$$= \frac{OA}{\pi OA} \quad (\text{construction and Th. I})$$

$$= \frac{1}{\pi}$$

$$\therefore \angle AOB = \frac{180^\circ}{\pi}$$

$$= \text{constant.}$$

Note. Theorem II can be briefly stated thus: π radians = 180° .

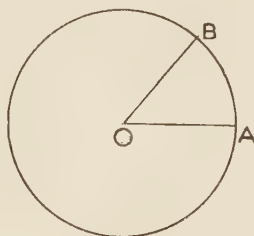


FIG. 2

Since the radian is a constant angle, it can be used as a unit. An angle expressed in radians is said to be expressed in circular measure.

Example 1. Express in circular measure $29^\circ 15' 37''$.

$$\begin{aligned} 29^\circ 15' 37'' &= 29.2603 \text{ (to four decimal places)} \\ &= \frac{29.2603 \times \pi}{180} \text{ radians} \\ &= \frac{29.2603 \times 3.1416}{180} \text{ radians} \\ &= 0.5107 \text{ radians (to four decimal places).} \end{aligned}$$

Example 2. Convert 2.5851 radians to sexagesimal measure.

$$\begin{aligned} 2.5851 \text{ radians} &= \frac{180^\circ \times 2.5851}{3.1416} \\ &= 148.11 \\ &= 148^\circ 7' \text{ (to nearest minute).} \end{aligned}$$

EXERCISES

(Nos. 1 to 5 should be answered orally.)

1. Express the following angles in radians :

$$30^\circ, 120^\circ, 210^\circ, 300^\circ, 360^\circ.$$

2. Express in degrees the angles whose circular measures are

$$\frac{1}{3}\pi, \frac{5}{6}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{3}{2}\pi, \frac{5}{3}\pi.$$

3. Two angles of a triangle are $\frac{\pi}{6}$ radians and $\frac{2\pi}{3}$ radians respectively.

Express the third angle in degrees.

4. What is (a) the complement, (b) the supplement, of $\frac{\pi}{6}$ radians? Give your answers in radians and degrees.

5. What is the ratio of 45° to $\frac{3}{4}\pi$ radians? $\frac{5}{9}\pi$ radians to $\frac{10}{9}\pi$ radians?

$\frac{1}{10}\pi$ radians to one right angle?

6. Convert $69^\circ 43' 57''$ into radians.

7. Express one radian in sexagesimal measure.

8. The sum of two angles is $\frac{11}{9}\pi$ radians, and their difference is $\frac{\pi}{8}$ radians. Find the number of degrees in each angle.

§ 3. THEOREMS CONCERNING CIRCULAR MEASURE

Theorem III. If an angle of θ radians is subtended at the centre of a circle of radius r by an arc of length l , $\theta = \frac{l}{r}$.

An arc of length r subtends at the centre an angle of 1 radian,
 $\therefore$ an arc of length l " " " $\frac{l}{r}$ radians,
i.e., $\theta = \frac{l}{r}$.

Theorem IV. The area of a sector of a circle is $\frac{1}{2}r^2\theta$, where r is the radius and θ radians is the angle contained by the sector.

Let arc AB of sector OADB (Fig. 3) be divided into n equal parts, and let CD be one of these parts. Join OC, OD, CD, and draw OM perpendicular to CD.

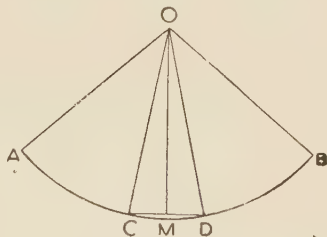


FIG. 3

Area of $\triangle OCD = \frac{1}{2}OM \times \text{chord } CD$
 and $n\triangle OCD = \frac{1}{2}OM \times nCD$.
 If n increases indefinitely,
 nCD tends to become the arc AB,
 OM " " radius,
 and
 $n\triangle OCD$ " " sector.

$$\begin{aligned}\therefore \text{area of sector} &= \frac{r}{2} \cdot \text{arc AB} \\ &= \frac{r}{2} \cdot r\theta \text{ (Th. III)} \\ &= \frac{1}{2}r^2\theta.\end{aligned}$$

Note. It follows from Th. IV that the area of a circle of radius r is

$$\frac{1}{2}r^2 2\pi \text{---i.e., } \pi r^2.$$

Example 1. Find, to the nearest degree, the angle subtended at the centre of a circle of radius 10 cm. by an arc of length 12 cm.

$$\begin{aligned}\text{Required angle} &= \frac{12}{10} \text{ radians (Th. III)} \\ &= \frac{12}{10} \times \frac{180^\circ}{\pi}\end{aligned}$$

$$\begin{aligned}
 &= \frac{12}{10} \times \frac{7}{22} \times 180^\circ \\
 &= 69^\circ \text{ (to nearest degree).}
 \end{aligned}$$

Example 2. A sector of a circle has area 45 sq. cm. and contains an angle of 30° . Find the radius of the circle, giving answer in cm., and correct to one decimal place.

Let length of radius = r cm.

$$\begin{aligned}
 \therefore \text{area of sector} &= \frac{1}{2} r^2 \times \frac{30\pi}{180} \text{ sq. cm. (Ths. II and IV)} \\
 &= \frac{\pi r^2}{12} \text{ sq. cm.}
 \end{aligned}$$

$$\therefore \frac{\pi r^2}{12} = 45$$

$$\begin{aligned}
 \therefore r^2 &= \frac{540}{\pi} \\
 &= 171.9
 \end{aligned}$$

$$\therefore r = 13.1$$

i.e., length of radius = 13.1 cm. (correct to one decimal place).

EXERCISES

1. The radius of a circle is 3.5 in. Find the area and perimeter of the circle.

2. The area of a circle is 55 sq. in. Find the length of the radius.

3. What length of wire can be bent into a circle of diameter 1.5 ft. ?

4. The perimeter of a semicircle is 43.2 cm. Find its area.

5. The area of a semicircle is 14.3 sq. ft. Find the perimeter.

6. Calculate the number of degrees in the angle subtended at the centre of a circle of radius 6 cm. by an arc of length 12 cm.

7. The circumference of a circle is 198 in. What length of arc subtends an angle of 25° at the centre ?

8. An angle of 42° is subtended at the circumference of a circle by an arc 8 cm. long. Find the radius of the circle.

9. In a circle whose diameter is 27 cm. an arc subtends an angle of $61^\circ 15'$ at the circumference. Find the length of the arc.

10. From two circles of radii 9.3 cm. and 12.4 cm. respectively two sectors whose angles are $\frac{\pi}{3}$ and $\frac{3\pi}{2}$ radians respectively are cut out. Find the ratio of the areas of the sectors.

11. A circular plot of diameter 15 ft. is divided into five equal sectors. Find the area of a sector.

12. A circular disc of radius 9 in. is divided into three equal sectors coloured red and three equal sectors coloured blue. The angle of a red sector is one half that of a blue sector. Find the area of each coloured sector.

13. In a circle of radius $5\sqrt{2}$ in. a chord of length 10 in. is placed. Find the area of the minor segment of the circle.

14. Find the number of radians in each angle of a regular pentagon, a regular octagon, and a regular decagon.

15. The angle subtended by the moon's diameter at the earth is 31 minutes. Find the diameter of the moon if its distance from the earth is 24×10^4 miles.

16. Find the distance between two places on the same meridian if the difference in latitude is $17^\circ 30'$. The earth's radius is 4000 miles.

17. A train moves round a circular curve of radius 600 ft. at 18 miles per hour. What angle is subtended at the centre of the circle by the arc traversed in 10 seconds?

18. ABC is an equilateral triangle whose sides are $2a$ units long. Find the area of the inscribed circle and show that the area of the circumscribed circle is four times that of the inscribed circle.

19. ABC is an isosceles right-angled triangle having $AB = AC = 2a$. Circles are described on AB and AC as diameters. Prove that the area of the part of the triangle common to both circles is $a^2 \left(\frac{\pi}{2} - 1 \right)$.

20. The radius of a circle, centre O , is a units long. OX and OZ are two radii, containing an angle of 120° . At X and Z tangents XY and ZY are drawn to the circle. Assuming that YO is $2a$ units, find the area enclosed by YX , YZ , and the major arc XZ .

CHAPTER II

CIRCULAR FUNCTIONS OF ANGLES

§ 4. RECTANGULAR AXES OF COORDINATES

Let XOX' , YOY' (Fig. 4), be two mutually perpendicular straight lines. From P , any point in the same plane as these lines, draw PM , PN , perpendicular to XX' , YY' , respectively. Let OM be considered positive or negative according as M is or is not on the same side of O as X , and let ON be considered positive or negative according as

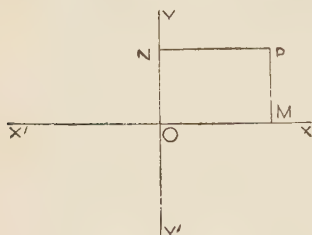


FIG. 4

N is or is not on the same side of O as Y . Then the position of P is known when the magnitudes and signs of OM and ON are known.

If OM and ON be x and y units respectively, x and y being given their proper signs, P is referred to as the point (x, y) . x is called the abscissa, y the ordinate, x and y the coordinates of the point P . XOX' , or briefly OX , is called the axis of x or simply the x -axis, and, in the same way, OY is called the y -axis, while OX and OY are called rectangular axes of coordinates, and O is called the origin.

Note 1. $ON = MP$, and hence MP is often referred to as the ordinate of the point P .

Note 2. The four portions into which the axes divide the plane are called quadrants, and these are referred to by number, as indicated in Fig. 5.

Note 3. The quadrant in which a point lies is known when the signs of its co-ordinates are known; e.g., a point which has a negative abscissa

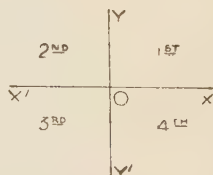


FIG. 5

and a positive ordinate must lie in the second quadrant. The signs of the coordinates of points in the different quadrants are shown in Fig. 6.

EXERCISE

Plot the points (2,8), (-2,0), (0,4), (1,6), (-4,-4), (-5,-6), and verify that they lie on a straight line.

	Y	
abscissa -		abscissa +
ordinate +		ordinate +
	X'	X
abscissa -		abscissa +
ordinate -		ordinate -
	Y'	

FIG. 6

§ 5. DEFINITION OF THE CIRCULAR FUNCTIONS

Consider rectangular axes OX, OY (Figs. 7 and 8). Let OP be the final position of a line which rotates from initial position OX through angle A, in the counter-clockwise direction if A is positive, and in the clockwise direction if A is negative.

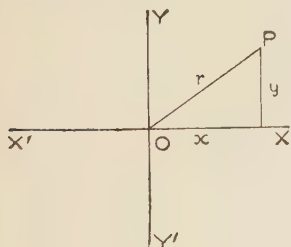


FIG. 7

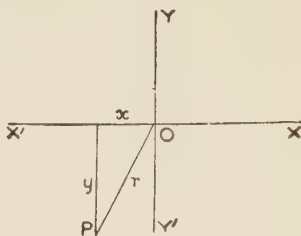


FIG. 8

Then, considering OP as a positive line of length r , and letting the coordinates of P be x and y ,

$$\text{sine } A = \frac{y}{r}, \quad \text{cosecant } A = \frac{r}{y},$$

$$\text{cosine } A = \frac{x}{r}, \quad \text{secant } A = \frac{r}{x},$$

$$\text{tangent } A = \frac{y}{x}, \quad \text{cotangent } A = \frac{x}{y};$$

or, briefly,

$$\sin A = \frac{y}{r}, \quad \text{cosec } A = \frac{r}{y},$$

$$\cos A = \frac{x}{r}, \quad \sec A = \frac{r}{x},$$

$$\tan A = \frac{y}{x}, \quad \cot A = \frac{x}{y}.$$

Note 1. These six ratios are called the *circular functions* of the angle **A**. Circular functions are sometimes called *trigonometrical functions* or *trigonometrical ratios*.

Note 2. A common cause of error among students beginning trigonometry is failure to realize that

the circular functions are ratios; the student must bear in mind that $\sin A$ is not the product of two quantities $\sin$ and **A**, but is a symbol for a ratio.

Note 3. Each of the circular functions is constant so long as the angle **A** is constant; i.e., the values of the ratios are independent of which point **P** is taken on the arm of the angle **A**,

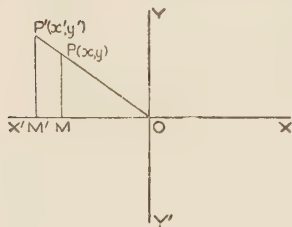


FIG. 9

e.g. (Fig. 9) let $P'(x', y')$ be any point on OP , or OP produced, and let $OP' = r'$. Draw MP , $M'P'$, the ordinates of P, P' respectively.

$$\text{Then } \sin A = \frac{y}{r} = \frac{MP}{OP} = \frac{M'P'}{OP'} \quad (\Delta's OMP, OM'P' \text{ are equiangular})$$

$$\text{i.e., } \sin A = \frac{y'}{r'}.$$

Similarly for the other circular functions.

Note 4. In the foregoing definitions the angle **A** is not necessarily less than four right angles; there is no limit to the magnitude which **A** may be given. It follows that any number of angles can be found having the same arms

OX and **OP**.

Note 5. The angle **A** is said to be in the first, second, third, or fourth quadrant according as the arm **OP** is in the first, second, third, or fourth quadrant. The signs of the circular functions of an angle are known, therefore, when we know in which quadrant the angle is. The student should verify the signs indicated in Fig. 10.

	sin +		sin +
	cos -		cos +
	tan -		tan +
x'		O	x
sin -			sin -
cos -			cos +
tan +			tan -
		y'	

FIG. 10

The cosecant, secant, and cotangent have been omitted in Fig. 10, as these functions have the same signs as their respective reciprocals, viz., sine, cosine, and tangent.

Example 1. Determine, by measurement, $\sin -141^\circ$.

Draw rectangular axes $X'OX$, $Y'OY$ (Fig. 11). Draw OA in the third quadrant, making $\hat{AOY}' 51^\circ$; then OX and OA are the arms of the angle -141° . From OA cut off OP 10 cm. long. Draw MP , the ordinate of P .

On measurement MP is found to be 6.3 cm. long.

$$\therefore \sin -141^\circ = \frac{-6.3}{10.0} = -0.63.$$

Note 1. In determining $\sin -141^\circ$ the second term in the ratio, viz., the line OP , was chosen of length 10 cm. This made the final calculation exceedingly simple, and at the same time allowed the value of

$\sin -141^\circ$ to be determined to two decimal places, because the length of MP could be stated to two significant figures.

Note 2. There is no necessity to draw the axes, because, having determined in which quadrant the angle is, we know the signs of its different circular functions. Except, however, in the case of angles in the first quadrant, where all the functions are positive, the student is advised to draw the axes.

Example 2. Determine, by measurement, $\tan 317^\circ$.

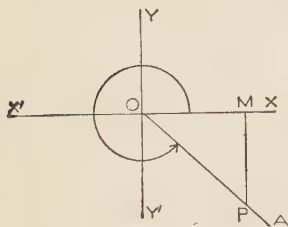


FIG. 12

Draw rectangular axes OX , OY (Fig. 12). Draw OA in the fourth quadrant, making $\hat{AOX} 43^\circ$; then OX and OA are the arms of the angle 317° . From OX cut off OM , 10 cm. long. Draw MP perpendicular to OA and meeting OA at P .

On measurement MP is found to be 9.3 cm. long.

$$\therefore \tan 317^\circ = \frac{-9.3}{10.0} = -0.93.$$

Example 3. Determine, by measurement, $\operatorname{cosec} 573^\circ$.

Draw rectangular axes $X'OX$, $Y'OY$ (Fig. 13). Draw OA in the

third quadrant, making $\hat{X'OA} 33^\circ$; then OX and OA are the arms of the angle 573° . From OY' cut off ON , 10.0 cm. long. Draw NP perpendicular to OY' and meeting OA at P ; draw MP , the ordinate of P . Then $MP=ON=10.0$ cm., and on measurement OP is found to be 18.4 cm. long. $\therefore \operatorname{cosec} 573^\circ = \frac{18.4}{-10.0} = -1.84$.

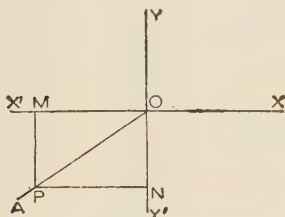


FIG. 13

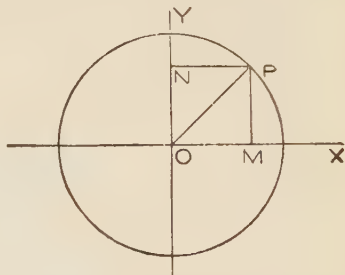


FIG. 14

Example 4. Determine, by measurement, the acute angle whose sine is 0.67.

Draw rectangular axes OX , OY (Fig. 14). Draw a circle, centre O and radius 10.0 cm. Along OY measure ON 6.7 cm. long. Draw NP parallel to OX and meeting the circumference of the circle at P . Join OP and draw MP , the ordinate of P .

Then, since $\sin \hat{XOP} = \frac{MP}{OP} = \frac{ON}{OP} = \frac{6.7}{10.0} = 0.67$, $\hat{XOP}$ is the required angle.

On measurement $\hat{XOP}$ is found to be 42° .

Example 5. Determine, by measurement, the angle which is greater than 270° but less than 360° , and whose tangent is -0.32 .

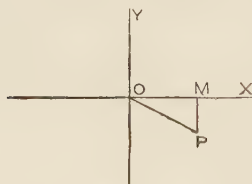


FIG. 15

Draw rectangular axes OX , OY (Fig. 15). Along OX measure OM , 10.0 cm. long; draw ordinate MP in the fourth quadrant and 3.2 cm. long. Join OP .

Then, since $\frac{MP}{OM} = \frac{-3.2}{10.0} = -0.32$, OX

and OP are the arms of the required angle. On measurement the acute angle XOP is found to be 18° . $\therefore$ the required angle is 342° .

Example 6. Determine, by measurement, the angle which is greater than 180° but less than 270° , and whose cosecant is -1.44 .

Draw rectangular axes $X'OX$, $Y'OY$ (Fig. 16). Draw a circle, centre O and radius 14.4 cm. Along OY' measure ON , 10.0 cm.

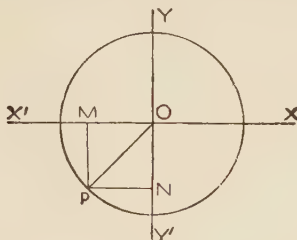


FIG. 16

long; draw NP parallel to OX' and meeting the circumference of the circle at P . Draw MP , the ordinate of P .

Then, since $\frac{OP}{MP} = \frac{14.4}{-10.0} = -1.44$, OX and OP are the arms of the required angle. On measurement the acute angle $X'OP$ is found to be 44° . $\therefore$ the required angle is 224° .

EXERCISES

Determine by measurement

- | | | |
|-----------------------|--|-----------------------|
| 1. $\sin 33^\circ$. | 3. $\tan 115^\circ$. | 5. $\sec 437^\circ$. |
| 2. $\cos 234^\circ$. | 4. $\operatorname{cosec} 302^\circ$. | 6. $\cot 835^\circ$. |
| 7. the angle, between | 0° and 90° , whose $\sin$ | is 0.34 . |
| 8. " " " | 90° " 180° , " $\cos$ | " -0.92 . |
| 9. " " " | 270° " 360° , " $\tan$ | " -3.27 . |
| 10. " " " | 180° " 270° , " cosec | " -1.37 . |
| 11. " " " | 270° " 360° , " $\sec$ | " 2.19 . |
| 12. " " " | 180° " 270° , " $\cot$ | " 1.23 . |

CHAPTER III

THE CIRCULAR FUNCTIONS OF CERTAIN ANGLES

§ 6. THE CIRCULAR FUNCTIONS OF 0° , 30° , 45° , AND MULTIPLES OF 30° , 45° .

Theorem I. If $\triangle ABC$ (Fig. 17) has $\hat{A} = 45^\circ$ and $\hat{B} = 90^\circ$ then

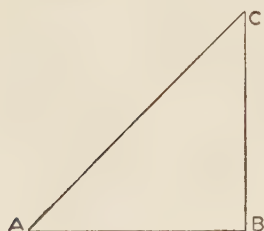


FIG. 17

$$AB : BC : CA = 1 : 1 : \sqrt{2}.$$

Let $AB = x$.

Then $BC = x$ ($\hat{C} = 45^\circ$)

and $CA^2 = 2x^2$ (Pythagoras' Th).

$$\therefore CA = \sqrt{2}x$$

$$\begin{aligned} \therefore AB : BC : CA &= x : x : \sqrt{2}x \\ &= 1 : 1 : \sqrt{2}. \end{aligned}$$

Theorem II. If $\triangle ABC$ (Fig. 18) has $\hat{A} = 60^\circ$ and $\hat{B} = 90^\circ$ then $AB : BC : CA = 1 : \sqrt{3} : 2$.

Produce AB to D , making $BD = AB$. Join CD .

Then $\triangle$'s ABC , DBC are congruent (two sides and included angle)

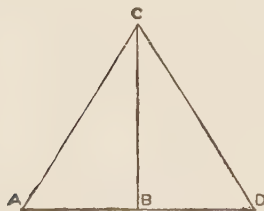


FIG. 18

$$\therefore \hat{D} = 60^\circ (\hat{D} = \hat{A})$$

$$\therefore \hat{ACD} = 60^\circ \text{ (3 angles of } \triangle)$$

$$\therefore AD = AC \text{ (sides opposite equal angles)}$$

$$\therefore AB = \frac{1}{2} AC$$

Let $AB = x$

Then $AC = 2x$

$$\begin{aligned} \therefore BC^2 &= 4x^2 - x^2 \text{ (Pythagoras' Theorem)} \\ &= 3x^2 \end{aligned}$$

$$\therefore BC = \sqrt{3}x$$

$$\begin{aligned}\therefore AB : BC : CA &= x : \sqrt{3}x : 2x \\ &= 1 : \sqrt{3} : 2.\end{aligned}$$

Note.— $\hat{ACB} = 30^\circ$ (Fig. 18), and if the angles of $\triangle ACB$ are taken in ascending order of magnitude then the corresponding sides are also in ascending order of magnitude.

By means of these theorems the circular functions of 30° , 45° , and certain multiples of 30° , 45° , can be determined.

Example 1. $\tan 30^\circ$.

OX and OA (Fig. 19) are the arms of the angle 30°

$$\begin{aligned}\therefore \tan 30^\circ &= \frac{MP}{OM} \\ &= \frac{1}{\sqrt{3}} \text{ (Th. II).}\end{aligned}$$

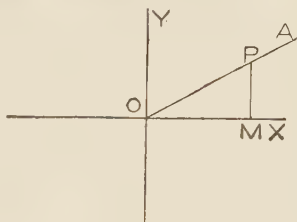


FIG. 19

Example 2. $\operatorname{cosec} 240^\circ$.

OX and OA (Fig. 20) are the arms of the angle 240°

and $\hat{MOP} = 60^\circ$

$$\begin{aligned}\therefore \operatorname{cosec} 240^\circ &= \frac{OP}{MP} \\ &= \frac{2}{-\sqrt{3}} \text{ (Th. II)} \\ &= -\frac{2}{\sqrt{3}}.\end{aligned}$$

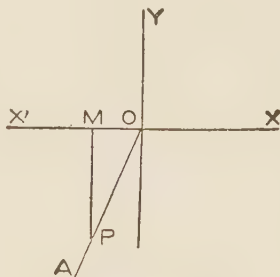


FIG. 20

Example 3. $\cos 315^\circ$.

OX and OA (Fig. 21) are the arms of the angle 315°

and $\hat{POM} = 45^\circ$

$$\begin{aligned}\therefore \cos 315^\circ &= \frac{OM}{OP} \\ &= \frac{1}{\sqrt{2}} \text{ (Th. I).}\end{aligned}$$

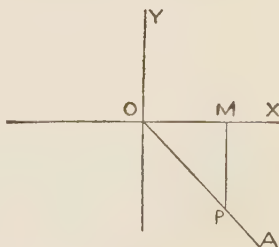


FIG. 21

In the case of the angles 0° , 90° , and multiples of 90° , the $\triangle OMP$ (as in Fig. 19, etc.) vanishes, but if the triangle is pictured as an infinitely narrow one the values of the ratios of these angles can be determined.

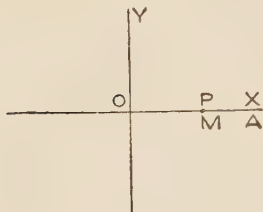


FIG. 22

Example 4. $\sin 0^\circ$.

OA (Fig. 22) coincides with OX

$$\begin{aligned}\therefore \sin 0^\circ &= \frac{MP}{OP} \\ &= \frac{0}{OP} \\ &= 0.\end{aligned}$$

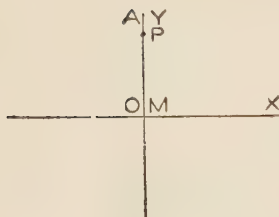


FIG. 23

Example 5. $\tan 90^\circ$.

OA (Fig. 23) coincides with OY

$$\begin{aligned}\therefore \tan 90^\circ &= \frac{MP}{OM} \\ &= \frac{MP}{0} \\ &= \infty.\end{aligned}$$



FIG. 24

Example 6. $\operatorname{cosec} 270^\circ$.

OA (Fig. 24) coincides with OY'

$$\begin{aligned}\therefore \operatorname{cosec} 270^\circ &= \frac{OP}{MP} \\ &= -1.\end{aligned}$$

By the methods of this section, the student should be able to determine any of the values in the following table, and after a little practice he should be able to do so mentally and quickly. The table is provided as an exercise, and no attempt should be made to memorize it.

θ	$\sin \theta$	$\cos \theta$	$\tan \theta$	$\operatorname{cosec} \theta$	$\sec \theta$	$\cot \theta$
0°	0	1	0	∞	1	∞
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$
90°	1	0	∞	1	∞	0
120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2	$-\frac{1}{\sqrt{3}}$
135°	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	$\sqrt{2}$	$-\sqrt{2}$	-1
150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$	$-\sqrt{3}$
180°	0	-1	0	∞	-1	∞
210°	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$	$\sqrt{3}$
225°	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	$-\sqrt{2}$	$-\sqrt{2}$	1
240°	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	-2	$\frac{1}{\sqrt{3}}$
270°	-1	0	∞	-1	∞	0
300°	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2	$-\frac{1}{\sqrt{3}}$
315°	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	$-\sqrt{2}$	$\sqrt{2}$	-1
330°	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$	$-\sqrt{3}$

Example 7. Find the value of

$$\sin 120^\circ \cot 330^\circ + \cos 210^\circ \tan 150^\circ + \operatorname{cosec} 225^\circ \sec 315^\circ.$$

$$\text{Expression} = \frac{\sqrt{3}}{2}(-\sqrt{3}) - \frac{\sqrt{3}}{2}\left(-\frac{1}{\sqrt{3}}\right) - \sqrt{2} \cdot \sqrt{2}$$

$$= -\frac{3}{2} + \frac{1}{2} - 2$$

$$= -3.$$

Example 8. Prove that

$$\cos \frac{2\pi}{3} \cos \frac{5\pi}{6} - \sin \frac{2\pi}{3} \sin \frac{5\pi}{6} = \cos \frac{5\pi}{3} \cos \frac{\pi}{6} + \sin \frac{5\pi}{3} \sin \frac{\pi}{6}.$$

$$\cos \frac{2\pi}{3} \cos \frac{5\pi}{6} - \sin \frac{2\pi}{3} \sin \frac{5\pi}{6} = -\frac{1}{2} \left(-\frac{\sqrt{3}}{2} \right) - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0$$

$$\cos \frac{5\pi}{3} \cos \frac{\pi}{6} + \sin \frac{5\pi}{3} \sin \frac{\pi}{6} = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{3}}{2} \right) \cdot \frac{1}{2} = 0$$

$$\therefore \cos \frac{2\pi}{3} \cos \frac{5\pi}{6} - \sin \frac{2\pi}{3} \sin \frac{5\pi}{6} = \cos \frac{5\pi}{3} \cos \frac{\pi}{6} + \sin \frac{5\pi}{3} \sin \frac{\pi}{6}.$$

Example 9. Prove that

$$\tan 4A = \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A} \text{ when } A = 30^\circ.$$

$$\text{If } A = 30^\circ, \tan 4A = \tan 120^\circ = -\sqrt{3}$$

$$\begin{aligned} \text{and } \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A} &= \frac{\frac{4}{\sqrt{3}} - \frac{4}{3\sqrt{3}}}{1 - 2 + \frac{1}{9}} \\ &= \frac{\frac{8}{3\sqrt{3}}}{-\frac{8}{9}} = -\sqrt{3}, \end{aligned}$$

$$\therefore \tan 4A = \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}.$$

EXERCISES

Find the values of

- $\sin 45^\circ \cos 45^\circ + \sin 30^\circ.$
- $\sin 60^\circ \cos 30^\circ - \cos 60^\circ \sin 30^\circ.$
- $\sin^2 30^\circ + \sin^2 60^\circ - \sin^2 90^\circ.$
- $\operatorname{cosec}^2 30^\circ \cot^2 45^\circ - \sin^2 45^\circ \tan^2 60^\circ.$
- $\cos^2 \frac{\pi}{6} - \cos^2 \frac{\pi}{3}.$
- $\cos \frac{\pi}{3} \sin \frac{\pi}{3} + \cos \frac{\pi}{6} \sin \frac{\pi}{6}.$
- $\sin \frac{\pi}{3} \tan \frac{\pi}{4} \tan \frac{\pi}{6} - \cos \frac{\pi}{6} \cot \frac{\pi}{4} \cot \frac{\pi}{3}.$
- $\sec^2 \frac{\pi}{6} \cot^2 \frac{\pi}{3} - 8 \cos^2 \frac{\pi}{4} \sin^2 \frac{\pi}{3}.$
- $\sin 240^\circ \cos 150^\circ - \sin 150^\circ \cos 240^\circ.$

10. $-\tan 300^\circ \tan 135^\circ \tan 240^\circ - 3 \cot 120^\circ \sec 210^\circ \operatorname{cosec} 330^\circ$.
11. $\sin^2 120^\circ \operatorname{cosec} 270^\circ - \cos^2 315^\circ \sec 180^\circ - \tan^2 225^\circ \cot 315^\circ$.
12. $3 \tan 210^\circ \sec 210^\circ - \sin 330^\circ \cot 135^\circ - \cos 150^\circ \operatorname{cosec} 240^\circ$.

$$13. \sin^2 \frac{2\pi}{3} + \cos^2 \frac{4\pi}{3}.$$

$$14. \operatorname{cosec} \frac{5\pi}{6} + \sec \frac{2\pi}{3} - \cot \frac{5\pi}{4}.$$

$$15. \cot^2 \frac{11\pi}{6} + \sin^2 \frac{3\pi}{2} - \operatorname{cosec}^2 \frac{7\pi}{6}.$$

$$16. \sec \frac{7\pi}{6} \tan \frac{5\pi}{3} - \cos \frac{5\pi}{6} \cot \frac{4\pi}{3} - \sin \frac{3\pi}{2} \operatorname{cosec} \frac{11\pi}{6}.$$

Prove that

17. $\cos 60^\circ + \cos 120^\circ - \cos 180^\circ = 1 - 4 \sin 30^\circ \sin 60^\circ \cos 90^\circ$.
18. $\tan 60^\circ + \tan 120^\circ + \tan 180^\circ = \cot 210^\circ \cot 330^\circ \tan 180^\circ$.
19. $(\sin 150^\circ + \cos 270^\circ + \tan 315^\circ)^2 = \sin^2 135^\circ \cos^2 225^\circ$.
20. $\sin^4 120^\circ + \cos^4 120^\circ = 1 - 2 \sin^2 120^\circ \cos^2 120^\circ$.
21. $\cos^2 45^\circ \cos^2 150^\circ - \sin^2 45^\circ \sin^2 150^\circ = (\cos^2 45^\circ - \sin^2 120^\circ) \tan 135^\circ$.
22. $\frac{\sin 120^\circ}{\tan 300^\circ} - \frac{\cos 240^\circ}{\cot 315^\circ} = \tan^2 240^\circ - \operatorname{cosec}^2 330^\circ$.
23. $\tan \frac{\pi}{3} \tan \frac{7\pi}{4} - \tan \frac{5\pi}{6} \tan \frac{4\pi}{3} + \tan \frac{7\pi}{6} \tan \frac{5\pi}{3} = \sin \frac{4\pi}{3} \sec \frac{2\pi}{3} \operatorname{cosec} \frac{3\pi}{2}$.
24. $\left(2 \cos^2 \frac{2\pi}{3} - 1\right) \left(1 - 2 \sin^2 \frac{5\pi}{3}\right) = 1 - \sin^2 \frac{4\pi}{3}$.

Verify the following identities :

25. $\sin 2\theta = 2 \sin \theta \cos \theta$
(i) when $\theta = 30^\circ$, (ii) when $\theta = 150^\circ$, (iii) when $\theta = 315^\circ$.
26. $\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$
(i) when $\theta = 135^\circ$, (ii) when $\theta = 180^\circ$, (iii) when $\theta = 240^\circ$.
27. $\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$
(i) when $\theta = 60^\circ$, (ii) when $\theta = -60^\circ$, (iii) when $\theta = 135^\circ$.
28. $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$
(i) when $\theta = 30^\circ$, (ii) when $\theta = 90^\circ$, (iii) when $\theta = 150^\circ$.
29. $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$
(i) when $\theta = 120^\circ$, (ii) when $\theta = 225^\circ$, (iii) when $\theta = 180^\circ$.
30. $\sin (A + B) = \sin A \cos B + \cos A \sin B$
(i) when $A = 30^\circ$, $B = 60^\circ$, (ii) when $A = 150^\circ$, $B = 30^\circ$.
31. $\cos (A + B) = \cos A \cos B - \sin A \sin B$
(i) when $A = 45^\circ$, $B = 90^\circ$, (ii) when $A = 120^\circ$, $B = 150^\circ$.

$$32. \sin (A - B) = \sin A \cos B - \cos A \sin B$$

(i) when $A = 180^\circ$, $B = 45^\circ$, (ii) when $A = 300^\circ$, $B = 240^\circ$.

$$33. \cos (A - B) = \cos A \cos B + \sin A \sin B$$

(i) when $A = 60^\circ$, $B = 30^\circ$, (ii) when $A = 270^\circ$, $B = 210^\circ$.

$$34. \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

(i) when $A = 150^\circ$, $B = 60^\circ$, (ii) when $A = 225^\circ$, $B = 45^\circ$.

$$35. \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

(i) when $A = 330^\circ$, $B = 210^\circ$, (ii) when $A = 240^\circ$, $B = 120^\circ$.

CHAPTER IV

RELATIONS BETWEEN THE CIRCULAR FUNCTIONS OF AN ANGLE

§ 7. RECIPROCAL RELATIONS

Take rectangular axes OX, OY (Fig. 25), and let OX and OA be the arms of the angle A . Let the coordinates of P , any point on OA , be x and y . Let $OP=r$.

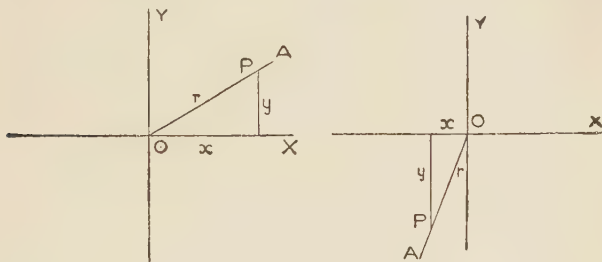


FIG. 25

$$\text{Then } \sin A = \frac{y}{r} = \frac{1}{\frac{r}{y}} = \frac{1}{\text{cosec } A}, \therefore \text{also cosec } A = \frac{1}{\sin A},$$

$$\cos A = \frac{x}{r} = \frac{1}{\frac{r}{x}} = \frac{1}{\sec A}, \therefore \text{also } \sec A = \frac{1}{\cos A},$$

$$\tan A = \frac{y}{x} = \frac{1}{\frac{x}{y}} = \frac{1}{\cot A}, \therefore \text{also } \cot A = \frac{1}{\tan A}.$$

§ 8. RELATIONS BETWEEN $\sin A$, $\cos A$, $\tan A$, AND $\cot A$

$$\text{In Fig. 25, } \tan A = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin A}{\cos A}, \text{ and } \therefore \text{also } \cot A = \frac{\cos A}{\sin A}$$

§ 9. RELATIONS DEPENDING UPON THE THEOREM OF PYTHAGORAS

In Fig. 25 we have, by the Theorem of Pythagoras, $y^2 + x^2 = r^2$; and this equation is true no matter what the signs of x and y may be, as only the squares of these numbers appear in the equation. We therefore have, for all values of A ,

$$y^2 + x^2 = r^2$$

$$\text{and } \therefore \frac{y^2}{r^2} + \frac{x^2}{r^2} = 1, \quad 1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}, \quad \text{and} \quad 1 + \frac{x^2}{y^2} = \frac{r^2}{y^2},$$

i.e., $\sin^2 A + \cos^2 A = 1$, $1 + \tan^2 A = \sec^2 A$, and $1 + \cot^2 A = \operatorname{cosec}^2 A$.

Note 1. These three identities are different forms of one identity: any two of them can be derived from the remaining one.

Note 2. The student should observe that $\sin^2 A$ or $(\sin A)^2$, but not $\sin A^2$, is the square of $\sin A$. The notation commonly used is $\sin^2 A$, and this should be read 'sin squared A .'

Example 1. Prove the identity $\operatorname{cosec} A - \sin A = \cos A \cot A$.

$$\begin{aligned} \operatorname{cosec} A - \sin A &= \frac{1}{\sin A} - \sin A \\ &= \frac{1 - \sin^2 A}{\sin A} \\ &= \frac{\cos^2 A}{\sin A} \\ &= \cos A \frac{\cos A}{\sin A} \\ &= \cos A \cot A. \end{aligned}$$

Example 2. Prove that $\tan A + \cot A - \operatorname{cosec} A = \frac{1 - \cos A}{\sin A \cos A}$.

$$\begin{aligned} \tan A + \cot A - \operatorname{cosec} A &= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A} - \frac{1}{\sin A} \\ &= \frac{\sin^2 A + \cos^2 A - \cos A}{\sin A \cos A} \\ &= \frac{1 - \cos A}{\sin A \cos A}. \end{aligned}$$

Example 3. Show that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ if $x = a(\sin a + \cos a)$ and

$$y = b(\sin a - \cos a).$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = (\sin a + \cos a)^2 + (\sin a - \cos a)^2$$

$$= \sin^2 a + 2 \sin a \cos a + \cos^2 a + \sin^2 a - 2 \sin a \cos a + \cos^2 a$$

$$= 2 (\sin^2 a + \cos^2 a)$$

$$= 2.$$

Example 4. If $\sec \phi = 2 \tan \phi$, show that $\sec^2 \phi = \frac{4}{3}$.

$$\begin{aligned}\sec^2 \phi &= 1 + \tan^2 \phi \\ &= 1 + \frac{1}{4} \sec^2 \phi.\end{aligned}$$

$$\therefore \frac{3}{4} \sec^2 \phi = 1$$

$$\therefore \sec^2 \phi = \frac{4}{3}.$$

EXERCISES

Prove that

1. $\sin A = \cos A \tan A.$
2. $\cos A = \sin A \cot A.$
3. $\tan \theta = \sin \theta \sec \theta.$
4. $\cot \phi = \cos \phi \operatorname{cosec} \phi.$
5. $(1 + \tan^2 a) \cos^2 a = 1.$
6. $\operatorname{cosec}^2 \beta \sin \beta \cos \beta = \cot \beta.$
7. $\cot^2 \beta \sec \beta = \cos \beta \operatorname{cosec}^2 \beta.$
8. $(\cos A + \cot A) \sec A = 1 + \operatorname{cosec} A.$
9. $\cot \theta (\sec^2 \theta - 1) = \tan \theta.$
10. $\frac{\sec a - 1}{\sec a + 1} = \frac{1 - \cos a}{1 + \cos a}.$
11. $(\sin \phi + \cos \phi)^2 = 1 + 2 \sin \phi \cos \phi.$
12. $(\cos \phi - \sin \phi)^2 = 1 - 2 \sin \phi \cos \phi.$
13. $\frac{\tan A}{1 - \tan^2 A} = \frac{\sin A \cos A}{\cos^2 A - \sin^2 A}.$
14. $\frac{\tan A}{1 + \tan^2 A} = \sin A \cos A.$
15. $\frac{\sin \theta}{\cot \theta \sec \theta} = 1 - \cos^2 \theta.$
16. $\frac{(1 - \sin A)(1 + \sin A)}{\sin^2 A} = \cot^2 A.$
17. $\frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$
18. $\frac{\sin \theta \cos \phi - \cos \theta \sin \phi}{\cos \theta \cos \phi + \sin \theta \sin \phi} = \frac{\cot \phi - \cot \theta}{1 + \cot \phi \cot \theta}.$
19. $\cot a + \tan a = \sec a \operatorname{cosec} a.$
20. $\operatorname{cosec}^4 A - 1 = \cot^2 A (\operatorname{cosec}^2 A + 1).$
21. $\frac{\cot A + \cot B}{\tan A + \tan B} = \cot A \cot B.$

$$22. \sin B + \cot B \cos B = \operatorname{cosec} B.$$

$$23. \cos^2 A (1 + \sin^2 A) = 1 - \sin^4 A.$$

$$24. \frac{\cos A}{1 + \sin A} = \sec A (1 - \sin A).$$

$$25. \frac{\sin B}{1 - \cos B} = \operatorname{cosec} B (1 + \cos B).$$

$$26. \frac{\cot \theta - \tan \theta}{\cot \theta + \tan \theta} = \cos^2 \theta - \sin^2 \theta.$$

$$27. \frac{1}{1 + \sin \phi} + \frac{1}{1 - \sin \phi} = 2 \sec^2 \phi.$$

$$28. \frac{1}{\sec A + \tan A} = \sec A - \tan A.$$

$$29. \sec A - \cos A = \sin A \tan A.$$

$$30. \frac{1 + \cot \theta}{1 + \tan \theta} = \cot \theta.$$

$$31. \cos B (\tan B - \cot B) = 2 \sin B - \operatorname{cosec} B.$$

$$32. 1 + \cot^2 \phi - 2 \sin^2 \phi - 2 \cos^2 \phi = \frac{2 \cos^2 \phi - 1}{\sin^2 \phi}.$$

$$33. \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta.$$

$$34. \frac{\sin A + \cos A}{\sin A - \cos A} = \frac{1 + 2 \sin A \cos A}{1 - 2 \cos^2 A}.$$

$$35. \cos^3 a + \sin^3 a = (\cos a + \sin a) (1 - \sin a \cos a).$$

$$36. \cos^4 a - \sin^4 a = \cos^2 a - \sin^2 a.$$

$$37. \sec^4 a - \tan^4 a = \sec^2 a (1 + \sin^2 a).$$

$$38. \operatorname{cosec}^4 A - \cot^4 A = \frac{1 + \cos^2 A}{\sin^2 A}.$$

$$39. \cos^6 \phi + \sin^6 \phi = 1 - 3 \sin^2 \phi \cos^2 \phi = 1 - 3 \sin^2 \phi + 3 \sin^4 \phi.$$

$$40. \cos^6 \theta - \sin^6 \theta = (1 - \sin^2 \theta \cos^2 \theta) (\cos^2 \theta - \sin^2 \theta).$$

$$41. \cos^8 \theta - \sin^8 \theta = (\cos^2 \theta - \sin^2 \theta) (1 - 2 \sin^2 \theta \cos^2 \theta).$$

$$42. \cos^8 A + \sin^8 A = 1 - 4 \sin^2 A \cos^2 A + 2 \sin^4 A \cos^4 A.$$

$$43. 2 \sin^2 a - \tan^2 a = \tan^2 a (1 - 2 \sin^2 a).$$

$$44. \frac{\cos A - \tan A \sin A}{\cos A + \tan A \sin A} = 1 - 2 \sin^2 A.$$

$$45. (\sin \theta + \cos \theta) (\sec \theta + \operatorname{cosec} \theta) = 2 + \tan \theta + \cot \theta.$$

$$46. (\sec^2 \theta - \tan^2 \theta) (\cos^2 \theta - \sin^2 \theta) = (1 - \sin^2 \theta) (1 - \tan^2 \theta).$$

$$47. \sin^2 A \cos^2 B - \cos^2 A \sin^2 B = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$$

$$48. \cos^2 \theta \cos^2 \phi - \sin^2 \theta \sin^2 \phi = \cos^2 \theta - \sin^2 \phi = \cos^2 \phi - \sin^2 \theta.$$

$$49. \sec A + \tan A + \cot A = \frac{1 + \sin A}{\sin A \cos A}.$$

$$50. \sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A.$$

$$51. \tan^2 A + \cot^2 A + 2 = \operatorname{cosec}^2 A \sec^2 A.$$

$$52. \frac{\tan A}{1 + \cot A} + \frac{\cot A}{1 + \tan A} = \frac{1 - \sin A \cos A}{\cos A \sin A}.$$

$$53. \frac{2 \cos^3 A - \cos A}{\sin A \cos^2 A - \sin^3 A} = \cot A.$$

$$54. \frac{1 - \sin A}{1 + \sin A} = 1 - 2 \sec A \tan A + 2 \tan^2 A.$$

$$55. (\tan A + \cot A)^2 + (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 5 + 2 \sec^2 A \operatorname{cosec}^2 A.$$

Show that

$$56. x^2 + y^2 = a^2 \text{ if } x = a \cos \theta, y = a \sin \theta.$$

$$57. \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ if } x = a \cos \theta, y = b \sin \theta.$$

$$58. x^2 - y^2 = c^2 \text{ if } x = c \sec \theta, y = c \tan \theta.$$

$$59. \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ if } x = a \sec \theta, y = b \tan \theta.$$

$$60. x^2 + y^2 + z^2 = r^2 \text{ if } x = r \cos \theta \sin \phi, y = r \sin \theta \sin \phi, z = r \cos \phi.$$

$$61. x^2 + y^2 = a^2 + b^2 \text{ if } x = a \cos \theta - b \sin \theta, y = a \sin \theta + b \cos \theta.$$

$$62. x^2 + y^2 - 2ax - 2ay + a^2 = 0 \text{ if } x = a(1 - \sin \theta), y = a(1 - \cos \theta).$$

$$63. xy + bx - ay = 2ab \text{ if } x = a(1 + \tan \theta), y = b(\cot \theta - 1).$$

$$64. x^2 + y^2 + 2gx + 2fy + c = 0 \text{ if } x = r \cos \theta - g, y = r \sin \theta - f, r^2 = g^2 + f^2 - c.$$

$$65. \text{ If } \frac{a}{\sin A} = \frac{b}{\cos A} \text{ show that } \sin A \cos A = \frac{ab}{a^2 + b^2}.$$

$$66. \text{ If } \frac{a}{\sec \theta} = \frac{b}{\tan \theta} \text{ show that } \sec \theta \tan \theta = \frac{ab}{a^2 - b^2}.$$

$$67. \text{ If } \frac{a+b}{\operatorname{cosec} \phi} = \frac{a-b}{\cot \phi} \text{ show that } \operatorname{cosec} \phi \cot \phi = \frac{a^2 - b^2}{4ab}.$$

$$68. \text{ If } \frac{a}{\sin A} = \frac{b}{\cos A} = \frac{c}{\tan A} \text{ show that } a^2(a^2 + b^2) = b^2c^2.$$

$$69. \text{ If } \sin^4 \theta + \cos^4 \theta = 1 \text{ show that either } \sin \theta = 0 \text{ or } \cos \theta = 0.$$

$$70. \text{ If } \sin \theta = \cos \theta \text{ show that } 2 \sin \theta \cos \theta = 1.$$

(The equations of § 53 may be taken at this stage.)

§ 10. ALL THE CIRCULAR FUNCTIONS OF AN ANGLE EXPRESSED IN TERMS OF ANY ONE OF THEM

Example 5. Express the circular functions of angle A in terms of $\sin A$.

$$\sin^2 A + \cos^2 A = 1$$

$$\therefore \cos A = \pm \sqrt{1 - \sin^2 A}$$

$$\text{and } \sec A = \frac{1}{\pm \sqrt{1 - \sin^2 A}}$$

$$\begin{aligned}
 \text{Also } \tan A &= \frac{\sin A}{\cos A} \\
 &= \frac{\sin A}{\pm \sqrt{1 - \sin^2 A}} \\
 \text{and } \cot A &= \frac{\pm \sqrt{1 - \sin^2 A}}{\sin A} \\
 \text{Also } \operatorname{cosec} A &= \frac{1}{\sin A}.
 \end{aligned}$$

Example 6. Express the circular functions of angle A in terms of $\tan A$.

$$\begin{aligned}
 \sec^2 A &= 1 + \tan^2 A \\
 \therefore \sec A &= \pm \sqrt{1 + \tan^2 A} \\
 \text{and } \cos A &= \frac{1}{\pm \sqrt{1 + \tan^2 A}} \\
 \text{Also } \sin A &= \cos A \tan A \\
 &= \frac{\tan A}{\pm \sqrt{1 + \tan^2 A}} \\
 \text{and } \operatorname{cosec} A &= \frac{\pm \sqrt{1 + \tan^2 A}}{\tan A} \\
 \text{Also } \cot A &= \frac{1}{\tan A}.
 \end{aligned}$$

Note 1. The student is familiar with the double solution in extracting a square root in algebra. In Example 5, if $\sin A$ is positive, A is in the first or second quadrant, and $\cos A$ is positive in the first quadrant and negative in the second quadrant. In Example 6, if $\tan A$ is negative, A is in the second or fourth quadrant, and $\sec A$ is negative in the second quadrant and positive in the fourth quadrant.

Note 2. The following method, in general use, enables us to obtain quickly the expressions connecting the circular functions, e.g., to express the circular functions of angle A in terms of $\tan A$.

Let $\tan A = t$, OM (Fig. 26) = 1 unit, $\angle OMP = 90^\circ$ and $MP = t$ units.

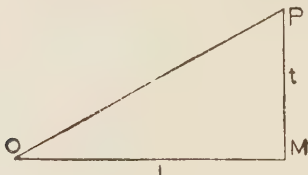


FIG. 26

$$\begin{aligned}\text{Then } OP &= \sqrt{OM^2 + MP^2} \\ &= \sqrt{1+t^2}\end{aligned}$$

$$\therefore \sin MOP = \frac{t}{\sqrt{1+t^2}}$$

$$\text{and } \cos MOP = \frac{1}{\sqrt{1+t^2}} \text{ etc.}$$

$\frac{t}{\sqrt{1+t^2}}$ and $\frac{1}{\sqrt{1+t^2}}$ are the numerical values of $\sin A$ and $\cos A$ respectively, in terms of $\tan A$; the sign of each circular function is determined from the quadrant in which A lies.

The student should use the method of this section to prove any of the values indicated in the table on page 36, but he may use the method of Note 2 to obtain them quickly. No attempt should be made to memorize any of the values: we have simply tabulated the results of a useful set of examples.

Example 7. If $\cos A = -\frac{1}{2}$ find the values of $\sin A$ and $\cot A$.

$$\begin{aligned}\sin A &= \pm \sqrt{1 - \cos^2 A} \\ &= \pm \sqrt{1 - \left(-\frac{1}{2}\right)^2} \\ &= \pm \frac{\sqrt{3}}{2}.\end{aligned}$$

If A is in the second quadrant $\sin A = \frac{\sqrt{3}}{2}$.

If A is in the third quadrant $\sin A = -\frac{\sqrt{3}}{2}$.

$$\begin{aligned}\cot A &= \frac{\cos A}{\sin A} \\ &= \frac{-\frac{1}{2}}{\pm \frac{\sqrt{3}}{2}} \\ &= \mp \frac{1}{\sqrt{3}}.\end{aligned}$$

If A is in the second quadrant $\cot A = -\frac{1}{\sqrt{3}}$.

If A is in the third quadrant $\cot A = \frac{1}{\sqrt{3}}$.

	Given $\sin A$	Given $\cos A$	Given $\tan A$	Given $\operatorname{cosec} A$	Given $\sec A$	Given $\cot A$
$\sin A$	$\sin A$	$\pm \sqrt{1 - \cos^2 A}$	$\frac{\tan A}{\pm \sqrt{1 + \tan^2 A}}$	$\frac{1}{\operatorname{cosec} A}$	$\frac{\pm \sqrt{\sec^2 A - 1}}{\sec A}$	$\frac{1}{\pm \sqrt{1 + \cot^2 A}}$
$\cos A$	$\pm \sqrt{1 - \sin^2 A}$	$\cos A$	$\frac{1}{\pm \sqrt{1 + \tan^2 A}}$	$\frac{\pm \sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A}$	$\frac{1}{\sec A}$	$\frac{\cot A}{\pm \sqrt{1 + \cot^2 A}}$
$\tan A$	$\frac{\sin A}{\pm \sqrt{1 - \sin^2 A}}$	$\frac{\pm \sqrt{1 - \cos^2 A}}{\cos A}$	$\tan A$	$\frac{1}{\pm \sqrt{\operatorname{cosec}^2 A - 1}}$	$\frac{\pm \sqrt{\sec^2 A - 1}}{\sec A}$	$\frac{1}{\cot A}$
$\operatorname{cosec} A$	$\frac{1}{\sin A}$	$\frac{1}{\pm \sqrt{1 - \cos^2 A}}$	$\frac{\pm \sqrt{1 + \tan^2 A}}{\tan A}$	$\operatorname{cosec} A$	$\frac{\sec A}{\pm \sqrt{\sec^2 A - 1}}$	$\frac{\pm \sqrt{1 + \cot^2 A}}{\cot A}$
$\sec A$	$\frac{1}{\pm \sqrt{1 - \sin^2 A}}$	$\frac{1}{\cos A}$	$\frac{\pm \sqrt{1 + \tan^2 A}}{\tan A}$	$\frac{\operatorname{cosec} A}{\pm \sqrt{\operatorname{cosec}^2 A - 1}}$	$\sec A$	$\frac{\pm \sqrt{1 + \cot^2 A}}{\cot A}$
$\cot A$	$\frac{\pm \sqrt{1 - \sin^2 A}}{\sin A}$	$\frac{\cos A}{\pm \sqrt{1 - \cos^2 A}}$	$\frac{1}{\tan A}$	$\pm \sqrt{\operatorname{cosec}^2 A - 1}$	$\frac{1}{\pm \sqrt{\sec^2 A - 1}}$	$\cot A$

EXERCISES

1. If $\cos A = \frac{4}{5}$, find the values of $\tan A$ and $\operatorname{cosec} A$.
2. If $\tan A = \frac{7}{24}$, find the values of $\sin A$ and $\cos A$.
3. If $\sin A = -\frac{9}{41}$, find the values of $\cos A$ and $\cot A$.
4. If $\sec A = -\frac{17}{15}$, find the values of $\operatorname{cosec} A$ and $\tan A$.
5. A lies between 180° and 270° and $\cot A = \frac{24}{7}$. Find the values of $\sin A$ and $\sec A$.
6. A lies between 270° and 360° and $\sec A = \frac{61}{60}$. Find the values of $\sin A$ and $\tan A$.
7. $\tan A = -\frac{1.3}{8.4}$. Find the positive value of $\cos A$ and hence determine by measurement the angle whose tangent and cosine have these values.
8. $\cos A = \frac{15}{17}$. Find the values of $\operatorname{cosec} A$ and state in which quadrant A lies if $\tan A$ is negative.
9. $\operatorname{cosec} A = -\frac{13}{12}$. Find the values of $\tan A$ and state in which quadrant A lies if $\sec A$ is negative.
10. If $\tan A$ is negative, find the value of $\sec A$ when $\sin A = -\frac{40}{41}$.
11. If A is an angle of a triangle, find the value of $\sin A$ when $\tan A = -\frac{14.4}{1.7}$.
12. Find by measurement, and to the nearest degree, the least positive value of angle A if $\tan A = \frac{6}{1.1}$ and $\cos A$ is negative.
13. $\sin A = 0.39$, and A lies between 90° and 180° . Assuming that $\sin 2A = 2 \sin A \cos A$, find the value of $\sin 2A$.
14. $\tan A = -0.74$. Assuming that $\cos 2A = \cos^2 A - \sin^2 A$, find the value of $\cos 2A$.
15. $\sin A = -0.68$ and A is in the third quadrant. Assuming that $\cos 3A = 4 \cos^3 A - 3 \cos A$, find the value of $\cos 3A$.
16. $\cos A = 0.91$ and A is in the fourth quadrant. Assuming that $\sin 3A = 3 \sin A - 4 \sin^3 A$, find the value of $\sin 3A$.

17. If $\cos A = \frac{1-t^2}{1+t^2}$, find the values of $\sin^2 A$ and $\tan^2 A$.
18. If $\tan A = \frac{2n(n+1)}{2n+1}$, find the values of $\sin^2 A$ and $\sec^2 A$.
19. If $\sin A = \frac{2\sqrt{ab}}{a+b}$, find the values of $\cos^2 A$ and $\cot^2 A$.

CHAPTER V

THE CIRCULAR FUNCTIONS OF $90^\circ \pm A$, $180^\circ \pm A$, $270^\circ \pm A$, AND $360^\circ - A$

§ 11. RELATIONS BETWEEN THE CIRCULAR FUNCTIONS OF ANY ANGLE A AND THOSE OF $90^\circ \pm A$, $270^\circ \pm A$

Example 1. $\cos (90^\circ - A) = \sin A$.

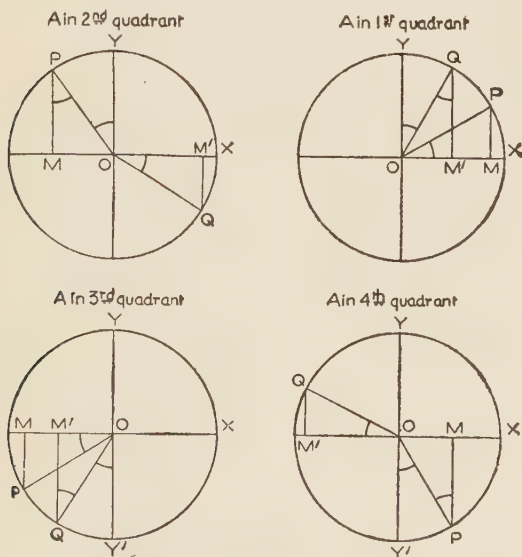


FIG. 27

Take rectangular axes OX , OY (Fig. 27). Let a line rotate through angle A from position OX to position OP , and let a line rotate through angle $90^\circ - A$ from position OX to position OQ . Take P and Q on a circle, centre O , and draw the ordinates MP , $M'Q$.

When **A** is in the first quadrant, $\hat{POM} = \hat{YOQ}$ (construction)
 $= \hat{OQM'}$ (**OY**, **M'Q** parallel).

When **A** is in the second quadrant, $\hat{M'ÔQ} = \hat{YÔP}$ (construction)
 $= \hat{MPO}$ (**OY**, **MP** parallel).

When **A** is in the third quadrant, $\hat{PÔM} = \hat{YÔQ}$ (construction)
 $= \hat{OQM'}$ (**OY'**, **M'Q** parallel).

When **A** is in the fourth quadrant, $\hat{M'ÔQ} = \hat{PÔY'}$ (construction)
 $= \hat{OPM}$ (**OY'**, **MP** parallel).

Also, in all cases, $\hat{OMP} = \hat{QM'O}$ (right angles)
 and $OP = OQ$ (construction)

$\therefore \triangle OMP \equiv \triangle QM'O$ (two angles and one side).

$$\therefore \cos (90^\circ - A) = \frac{OM'}{OQ} = \frac{MP}{OP} = \sin A.$$

Example 2. $\sec (270^\circ - A) = -\operatorname{cosec} A.$

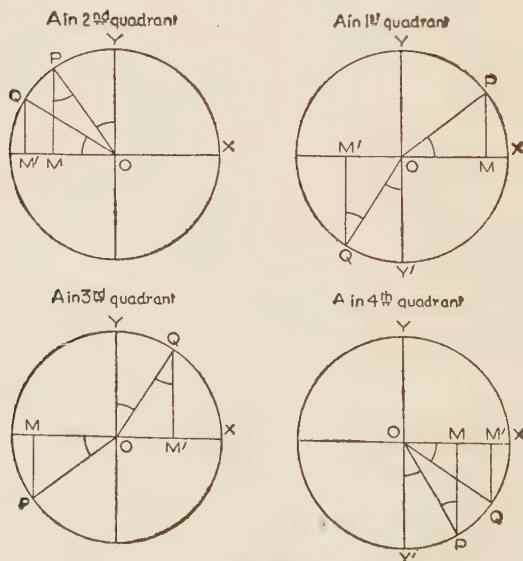


FIG. 28

In this case let $\hat{XOP} = A$, $\hat{XOQ} = 270^\circ - A$ (Fig. 28).

When A is in the first quadrant, $\hat{POM} = \hat{Y'OQ}$ (construction)
 $= \hat{OQM'}$ (OY' , $M'Q$ parallel).

When A is in the second quadrant, $\hat{M'OQ} = \hat{POY}$ (construction)
 $= \hat{MPO}$ (OY , MP parallel).

When A is in the third quadrant, $\hat{POM} = \hat{YOQ}$ (construction)
 $= \hat{M'QO}$ (OY , $M'Q$ parallel).

When A is in the fourth quadrant, $\hat{M'OQ} = \hat{POY'}$ (construction)
 $= \hat{MPO}$ (OY' , MP parallel).

Also, in all cases, $\hat{OMP} = \hat{QM'O}$ (right angles)
 and $OP = OQ$ (construction)

$\therefore \triangle OMP \equiv \triangle QM'O$ (two angles and one side).

$$\therefore \sec (270^\circ - A) = \frac{OQ}{OM'} = \frac{OP}{-MP} = -\operatorname{cosec} A.$$

All the circular functions of $90^\circ \pm A$, $270^\circ \pm A$ can be obtained in a similar manner, and the results are tabulated in § 12.

Note. The results of Examples 1 and 2 are readily obtained by taking angle A in the first quadrant. In the examples of the next section we shall consider A in the first quadrant, and leave the student to prove the results for all cases.

§ 12. RELATIONS BETWEEN THE CIRCULAR FUNCTIONS OF ANY ANGLE A AND THOSE OF $180^\circ \pm A$, $360^\circ - A$

Example 3. $\sin (180^\circ - A) = \sin A$.

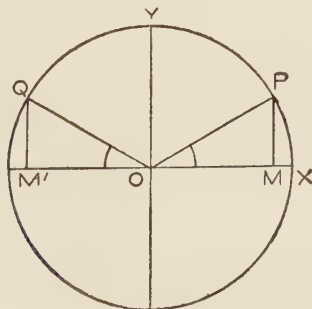


FIG. 29

In this case $\hat{XOP} = A$, $\hat{XOQ} = 180^\circ - A$ (Fig. 29).

$\triangle MOP \equiv \triangle M'OQ$ (two angles and one side)

$$\therefore \sin(180^\circ - A) = \frac{M'Q}{OQ} = \frac{MP}{OP} = \sin A.$$

Example 4. $\tan(360^\circ - A) = -\tan A$.

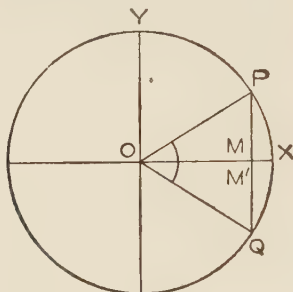


FIG. 30

In this case $\hat{XOP} = A$, $\hat{XOQ} = 360^\circ - A$ (Fig. 30).

$\triangle MOP \equiv \triangle M'OQ$ (two angles and one side)

$$\therefore \tan(360^\circ - A) = \frac{M'Q}{OM'} = \frac{-MP}{OM} = -\tan A.$$

All the circular functions of $180^\circ \pm A$, $360^\circ - A$ are obtained in a similar manner and are included in the following table. Again we remind the student that he should obtain any of the results as an exercise.

	$90^\circ - A$	$90^\circ + A$	$180^\circ - A$	$180^\circ + A$	$270^\circ - A$	$270^\circ + A$	$360^\circ - A$
sin	cos A	cos A	sin A	-sin A	-cos A	-cos A	-sin A
cos	sin A	-sin A	-cos A	-cos A	-sin A	sin A	cos A
tan	cot A	-cot A	-tan A	tan A	cot A	-cot A	-tan A
cosec	sec A	sec A	cosec A	-cosec A	-sec A	-sec A	-cosec A
sec	cosec A	-cosec A	-sec A	-sec A	-cosec A	cosec A	sec A
cot	tan A	-tan A	-cot A	cot A	tan A	-tan A	-cot A

Note 1. If we call sine and cosine co-functions, and likewise tangent and cotangent, secant and cosecant, the values in the preceding table may be recalled readily thus :

(i) Consider A in the first quadrant and find the quadrant in which lies the angle whose function is required: the sign of that function for angles in this quadrant is the required sign.

(ii) If the angle whose circular function is required is $180^\circ \pm A$ or $360^\circ - A$ take the corresponding function of A : if the angle whose circular function is required is $90^\circ \pm A$ or $270^\circ \pm A$ take the co-function of A .

The following example illustrates the method of recalling cosec $(270^\circ + A)$.

If A is in the first quadrant, $270^\circ + A$ is in the fourth quadrant. The cosecants of angles in the fourth quadrant are **negative**.

The angle considered is $270^\circ + A$, $\therefore$ the required function of A is the secant.

$\therefore$ the required value is $-\sec A$.

Note. 2. The position of OP in any of the diagrams of Figs. 27 and 28 is not altered when A changes in value by any multiple of 360° ; i.e., when n is any integer the position of OP is the same for $n360^\circ + A$ as for A . The values of the circular functions of $n360^\circ + A$ are therefore the same as those of corresponding circular functions of A .

Example 5. Express 785° as a circular function of an angle between 0° and 45° .

$$\begin{aligned}\sec 785^\circ &= \sec (2 \times 360^\circ + 65^\circ) \\ &= \sec 65^\circ \\ &= \operatorname{cosec} 25^\circ.\end{aligned}$$

Example 6. Express $\cot (-821^\circ)$ as in Example 5.

$$\begin{aligned}\cot (-821^\circ) &= \cot (3 \times 360^\circ - 821^\circ) \\ &= \cot 259^\circ \\ &= \cot (180^\circ + 79^\circ) \\ &= \cot 79^\circ \\ &= \tan 11^\circ.\end{aligned}$$

Example 7. Prove that

$$\begin{aligned}\operatorname{cosec} (270^\circ - A) \cot (360^\circ - A) \sin (90^\circ - A) \\ &= \cos (90^\circ + A) \tan (270^\circ + A) \operatorname{cosec} (180^\circ - A). \\ \operatorname{cosec} (270^\circ - A) \cot (360^\circ - A) \sin (90^\circ - A) \\ &= (-\sec A) (-\cot A) \cos A \\ &= \cot A \\ \cos (90^\circ + A) \tan (270^\circ + A) \operatorname{cosec} (180^\circ - A) \\ &= (-\sin A) (-\cot A) \operatorname{cosec} A \\ &= \cot A\end{aligned}$$

$$\begin{aligned}\therefore \operatorname{cosec} (270^\circ - A) \cot (360^\circ - A) \sin (90^\circ - A) \\ = \cos (90^\circ + A) \tan (270^\circ + A) \operatorname{cosec} (180^\circ - A).\end{aligned}$$

Example 8. If $A + B + C = 180^\circ$ prove that

$$\begin{aligned}2 \sin^2 \frac{B+C}{2} - 1 &= \sin^2 A + \cos^2 (B+C) - 2 \sin^2 \frac{A}{2} \\ \sin^2 A + \cos^2 (B+C) - 2 \sin^2 \frac{A}{2} &= \sin^2 A + \cos^2 (180^\circ - A) - 2 \sin^2 \frac{A}{2} \\ &= \sin^2 A + \cos^2 A - 2 \sin^2 \frac{A}{2} \\ &= 1 - 2 \sin^2 \left(90^\circ - \frac{B+C}{2} \right) \\ &= 1 - 2 \cos^2 \frac{B+C}{2} \\ &= 2 \sin^2 \frac{B+C}{2} - 1.\end{aligned}$$

EXERCISES

Find the values of

- | | | |
|---------------------------------------|--|---------------------------|
| 1. $\cos 405^\circ$. | 5. $\sin 660^\circ$. | 9. $\tan (-330^\circ)$. |
| 2. $\operatorname{cosec} 750^\circ$. | 6. $\sec (-225^\circ)$. | 10. $\cot 1290^\circ$. |
| 3. $\tan 960^\circ$. | 7. $\operatorname{cosec} 1050^\circ$. | 11. $\sin 570^\circ$. |
| 4. $\cot 1215^\circ$. | 8. $\cos 675^\circ$. | 12. $\sec (-855^\circ)$. |

Express as the function of an angle between 0° and 45°

- | | | |
|--|---------------------------|--|
| 13. $\sin 289^\circ$. | 17. $\sec 803^\circ$. | 21. $\operatorname{cosec} 658^\circ$. |
| 14. $\cot (-59^\circ)$. | 18. $\cos (-80^\circ)$. | 22. $\cos 1000^\circ$. |
| 15. $\tan 245^\circ$. | 19. $\cot (-583^\circ)$. | 23. $\sec (-1292^\circ)$. |
| 16. $\operatorname{cosec} 127^\circ$. | 20. $\sin 1135^\circ$. | 24. $\tan (n360^\circ + 70^\circ)$. |

Prove that

25. $\tan (90^\circ - A) \sec (180^\circ + A) \cos (90^\circ + A) = 1$.
26. $\sin (360^\circ - A) \operatorname{cosec} (180^\circ + A) \cot (270^\circ - A) = \cot (90^\circ - A)$.
27. $\tan (180^\circ - A) \sin (270^\circ + A) \operatorname{cosec} (360^\circ - A) = -1$.
28. $\cos (\pi + \theta) \cot \left(\frac{\pi}{2} + \theta \right) \sec (2\pi - \theta) = \tan (\pi + \theta)$.
29. $\operatorname{cosec} \left(\frac{3\pi}{2} + \theta \right) \sin \left(\frac{\pi}{2} - \theta \right) \cos (\pi - \theta) + \sin \left(\frac{3\pi}{2} - \theta \right) = 0$.
30. $\cot (2\pi - \theta) \sin (\pi - \theta) \operatorname{cosec} \left(\frac{\pi}{2} + \theta \right) + 1 = 0$.
31. $\sin (360^\circ - A) + \tan (270^\circ + A) - \sin (180^\circ + A) = \tan (90^\circ + A)$.
32. $\cos (180^\circ - A) - \sin (270^\circ - A) + \cos (90^\circ - A) - \cos (270^\circ + A) = 0$.
33. $\operatorname{cosec} (180^\circ + A) + \sec (90^\circ - A) - \cot (360^\circ - A) = \tan (270^\circ - A)$.

$$- 34. \sin(\pi + \theta) \sec\left(\frac{\pi}{2} - \theta\right) - \tan\left(\frac{3\pi}{2} + \theta\right) \cot\left(\frac{3\pi}{2} - \theta\right) = 0.$$

$$- 35. \sin^2(\pi - \theta) + \cos^2(2\pi - \theta) = 1.$$

$$- 36. \frac{\tan(\pi + \theta) + \cot\left(\frac{3\pi}{2} - \theta\right)}{1 + \tan(2\pi - \theta) \cot\left(\frac{\pi}{2} + \theta\right)} = 2 \sin \theta \cos \theta.$$

► If $A + B + C = 180^\circ$ prove that

$$37. \tan(B + C) = -\cot(90^\circ - A). \quad 41. \cos \frac{C + A}{2} = -\cos\left(90^\circ + \frac{B}{2}\right).$$

$$38. \sin(90^\circ + B) = -\cos(C + A).$$

$$39. \sec(A + B) = \sec(180^\circ + C). \quad 42. \sin \frac{A + B}{2} = \sin\left(90^\circ + \frac{C}{2}\right).$$

$$40. \cos(90^\circ + A + B) = -\sin(180^\circ - C). \quad 43. \tan \frac{B + C}{2} = -\tan\left(90^\circ + \frac{A}{2}\right).$$

$$44. \sin \frac{A + B + C}{2} = \tan \frac{A + B}{2} \tan \frac{C}{2}.$$

$$45. \sin(B + C) + \sin(90^\circ - A) \tan(B + C) = 0.$$

$$46. \cos(B + C) + \sin(B + C) \tan(90^\circ - A) = 0.$$

$$47. \sec(A + B) \cot(180^\circ - C) = \operatorname{cosec}(A + B).$$

$$48. \operatorname{cosec}(90^\circ - A) = -\sec(90^\circ - A) \tan(B + C).$$

$$49. \sin\left(90^\circ - \frac{C}{2}\right) = \cos \frac{A + B}{2} \tan\left(90^\circ - \frac{C}{2}\right).$$

$$50. \cos \frac{B + C}{2} = \sin\left(90^\circ - \frac{A}{2}\right) \cot \frac{B + C}{2}.$$

$$51. \operatorname{cosec} \frac{A + C}{2} \tan\left(90^\circ - \frac{B}{2}\right) = \operatorname{cosec}\left(180^\circ - \frac{B}{2}\right).$$

$$52. \sec \frac{A + B}{2} \cot\left(90^\circ - \frac{C}{2}\right) = \operatorname{cosec} \frac{A + B}{2}.$$

$$53. \cos A - \sin(B + C) \tan(B + C) = \sec A.$$

$$54. \frac{\cot \frac{B + C}{2}}{\sin \frac{A}{2} \sin \frac{B + C}{2}} = \frac{1 - \tan^2 \frac{A}{2}}{\sin^2 \frac{B + C}{2} - \sin^2 \frac{A}{2}}.$$

§ 13. TABULATED VALUES OF THE CIRCULAR FUNCTIONS

The values of the circular functions of any angle can be expressed in terms of those of an angle between 0° and 45° . In practice, however, we use tables which give the circular functions of angles from 0° to 90° .

The values of sine, cosine, and tangent are given at intervals of six minutes as indicated by the heading of each column. A difference table enables us to give these values for angles differing by one minute.

NATURAL SINE TABLE

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
65°	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6

Tables of natural sines, cosines, and tangents are printed at the end of the book. The above row is taken from the table of natural sines in which the columns are headed as shown. For angles between 65° and 66° the values of the sine are given at intervals of six minutes. When the angle is increased by 1, 2, 3, 4, or 5 minutes, the number in the corresponding difference column is added to the number already obtained;

$$\text{e.g., } \sin 65^\circ 42' = 0.9114$$

$$\sin 65^\circ 46' = 0.9119 \text{ (angle is increased by } 4', \text{ and 5 is added to } 9114).$$

$$\text{Note. } \cos 24^\circ 51' = \sin 65^\circ 9' \\ = 0.9074.$$

NATURAL COSINE TABLE

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
24°	·9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6

In the case of the cosine table, the number in the corresponding difference column is subtracted from the number already obtained;

$$\text{e.g., } \cos 24^\circ 18' = 0.9114$$

$$\cos 24^\circ 22' = 0.9109 \text{ (angle is increased by } 4', \text{ and 5 is subtracted from } 9114).$$

$$\text{Note. } \sin 65^\circ 16' = \cos 24^\circ 44' \\ = 0.9083.$$

NATURAL TANGENT TABLE

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
63°	1.9626	9711	9797	9883	9970	0057	0145	0233	0323	0413	15	29	44	58	73

In the case of the tangent table, the number in the corresponding difference column is added to the number already obtained;

$$\text{e.g., } \tan 63^\circ 24' = 1.9970$$

$$\tan 63^\circ 39' = 2.0189.$$

Note 1. $\tan 63^\circ 24' = 1.9970$ and $\tan 63^\circ 30' = 2.0057$. The integral parts here differ by one, and such a change is indicated in this and the other tables of this book by using heavier type. Sometimes the change is indicated by a short rule thus: $\bar{0}057$.

$$\text{Note 2. } \cot 26^\circ 13' = \tan 63^\circ 47' = 2.0306.$$

Note 3. The acute angle whose tangent is 2.0338 is found as follows. The required angle is greater than $63^\circ 48'$, whose tangent is 2.0323. If 15 is added to 0323 we get the required decimal figures, and this corresponds to an increase of $1'$ to $63^\circ 48'$. $\therefore$ required angle $= 63^\circ 49'$.

Note 4. The acute angle whose tangent is 1.9993 is $63^\circ 26'$ (to nearest minute) or $63^\circ 25\frac{1}{2}'$ (to nearest half-minute).

Note 5. The adjective *natural* is applied to circular functions when we wish to emphasize that we are not dealing with the logarithms of these functions (see p. 116).

Tables of cosecants, secants, and cotangents are not in common use: the values of cosecants and secants are obtained from the values of their reciprocals, while the values of cotangents may be obtained after the manner indicated in Note 2 above.

The student should test his ability to use tables by verifying that

$$1. \sin 8^\circ 36' = 0.1495.$$

$$2. \sin 107^\circ 33' = 0.9535.$$

$$3. \sin 28^\circ 59' = 0.4846.$$

$$4. \cos 37^\circ 57' = 0.7886.$$

$$5. \cos 81^\circ 53' = 0.1412.$$

$$6. \cos 243^\circ 3' = -0.4532.$$

$$7. \tan 41^\circ 13' = 0.8759.$$

$$8. \tan 53^\circ 29' = 1.3506.$$

$$9. \tan 71^\circ 49' = 3.0444.$$

$$10. \tan 323^\circ 17' = -0.7459.$$

$$11. \operatorname{cosec} 10^\circ 24' = 5.54.$$

$$12. \cot 213^\circ 19' = 1.5214.$$

$$13. \sec 74^\circ 21' = 3.71.$$

$$14. \operatorname{cosec} 67^\circ 59' = 1.08.$$

$$15. \cot 115^\circ 37' = -0.4795.$$

$$16. \sec 224^\circ 44' = -1.41.$$

CHAPTER VI

SOLUTION OF RIGHT-ANGLED TRIANGLES

§ 14. The three angles and the three sides of a triangle are known as the six elements of the triangle. If, given some of the elements, we determine the others, we are said to solve the triangle. In § 15 we shall show how to solve right-angled triangles, and in doing so we shall denote the three angles by the letters A, B, C , and the numbers of units of length in the sides opposite these angles by a, b, c respectively; the right angle will be denoted by C .

§ 15. The right-angled triangle can be solved if, excluding the right angle, we are given any two elements other than the two remaining angles; for the following cases arise:

- (A) Given the hypotenuse and a side,
e.g. (Fig. 31), given c and a ,



FIG. 31

B is obtained from the relation $\cos B = \frac{a}{c}$,

	A	"	"	"	"	"	$A = 90^\circ - B,$
$\frac{b}{a} = \tan B$, & $\therefore b$	"	"	"	"	"	"	$b = a \tan B.$

Note 1. The reader should accustom himself to establishing relations such as $b = a \tan B$ without having to write down the

intermediate step, viz., $\frac{b}{a} = \tan B$. Such intermediate steps will be omitted in cases (B), (C), (D).

Note 2. The relation $\sin A = \frac{a}{c}$ can also be used to determine A , and the relation $b = \sqrt{c^2 - a^2}$ can be used to determine b . Similar alternative methods can be used in cases (B), (C), (D).

Note 3. The method of solution is the same if c and b are the given elements; this case is left as an exercise to the student.

- (B) Given the two sides containing the right angle,
i.e. (Fig. 32), given a and b ,

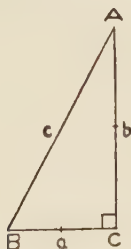


FIG. 32

A is obtained from the relation $\tan A = \frac{a}{b}$,

B " " " " " $B = 90^\circ - A$,

and c " " " " " $c = \frac{b}{\sin B}$.

- (C) Given the hypotenuse and an angle,
e.g. (Fig. 33), given c and B ,



FIG. 33

a is obtained from the relation $a = c \cos B$,
 b " " " " " $b = c \sin B$,
 and A " " " " " $A = 90^\circ - B$.

Note. The method of solution is the same if c and A are the given elements ; this case is left as an exercise to the student.

(D) Given one side and one angle,
e.g. (Fig. 34), given a and B ,

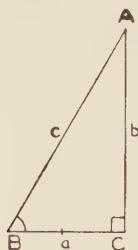


FIG. 34

A is obtained from the relation $A = 90^\circ - B$,
 b " " " " " $b = a \tan B$,
 and c " " " " " $c = \frac{a}{\cos B}$.

Note 1. If the given angle is opposite the given side the same method of solution is applicable, for the third angle, being the complement of the given angle, is known.

Note 2. The method of solution is the same if b and A are the given elements ; this case is left as an exercise to the student.

Note 3. Excepting the relations $A + B = 90^\circ$ and $c^2 = a^2 + b^2$, all the relations used in the preceding four cases follow immediately from the definitions of the circular functions of an angle, and in solving right-angled triangles the student should depend on his knowledge of the definitions of the circular functions and not on the memorizing of any of the formulæ appearing in the foregoing cases.

Example 1. Solve the triangle ABC , given that $C = 90^\circ$, $B = 60^\circ$, and $c = 10$.

$$\begin{aligned}
 \text{In Fig. 35, } a &= c \cos B \\
 &= 10 \cos 60^\circ \\
 &= 10 \times \frac{1}{2} \\
 &= 5
 \end{aligned}$$

$$\begin{aligned}
 b &= c \sin B \\
 &= 10 \sin 60^\circ \\
 &= 10 \times \frac{\sqrt{3}}{2} \\
 &= 5\sqrt{3}
 \end{aligned}$$

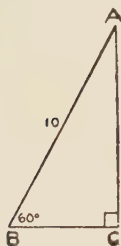


FIG. 35

$$\begin{aligned}
 \text{and } A &= 90^\circ - B \\
 &= 90^\circ - 60^\circ \\
 &= 30^\circ.
 \end{aligned}$$

Example 2. Solve the triangle ABC in which $B = 90^\circ$, $b = 13$, and $c = 5$.

$$\begin{aligned}
 \text{In Fig. 36, } \cos A &= \frac{5}{13} \\
 &= 0.3846 \\
 \therefore A &= 67^\circ 23'
 \end{aligned}$$

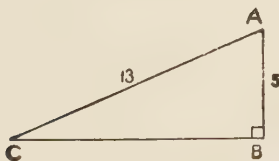


FIG. 36

$$\begin{aligned}
 C &= 90^\circ - A \\
 &= 90^\circ - 67^\circ 23' \\
 &= 22^\circ 37'
 \end{aligned}$$

$$\begin{aligned}
 a &= c \tan A \\
 &= 5 \tan 67^\circ 23' \\
 &= 5 \times 2.4005 \\
 &= 12.00 \text{ (to two decimal places).}
 \end{aligned}$$

Example 3. If in the triangle ABC , $A = 90^\circ$, $B = 43^\circ$, and $b = 15.7$, find c .

$$\begin{aligned}\text{In Fig. 37, } C &= 90^\circ - B \\ &= 90^\circ - 43^\circ \\ &= 47^\circ\end{aligned}$$

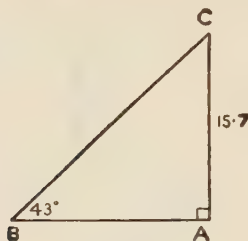


FIG. 37

$$\begin{aligned}c &= b \tan C \\ &= 15.7 \tan 47^\circ \\ &= 15.7 \times 1.0724 \\ &= 16.8 \text{ (to three significant figures).}\end{aligned}$$

EXERCISES

1. Solve the triangle ABC in which $a = 10$, $b = 10\sqrt{3}$, and $C = 90^\circ$.
2. " " " $b = 100$, $C = 60^\circ$, and $A = 90^\circ$.
3. " " " $c = 3.2$, $B = 57^\circ$, and $C = 90^\circ$.
4. " " " $b = 9.8$, $c = 4.7$, and $B = 90^\circ$.
5. " " " $a = 59$, $b = 95$, and $C = 90^\circ$.
6. " " " $b = 105$, $A = 90^\circ$, and $C = 83^\circ$.
7. Determine A when $a = 10$, $b = 3.7$, and $C = 90^\circ$.
8. " b " $c = 8.9$, $B = 43^\circ$, and $C = 90^\circ$.
9. " c " $b = 9.4$, $A = 90^\circ$, and $C = 67^\circ$.
10. In a right-angled triangle the sides containing the right angle have lengths 8 ft. and 11 in. respectively; determine the size of the smallest angle.

§ 16. Many simple problems can be solved by the methods of § 15. In the statement of such problems the phrases 'angle of elevation' and 'angle of depression' are often used; these phrases may be explained thus:

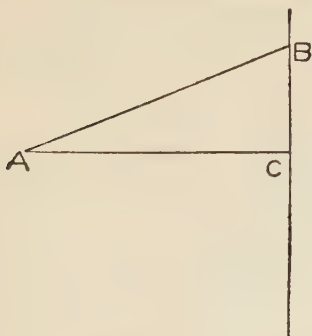


FIG. 38

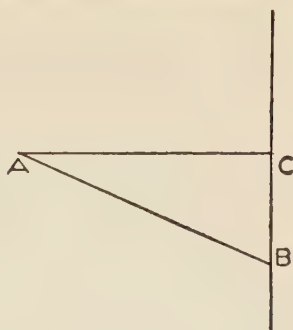


FIG. 39

Let A and B be any two points, and let AC be perpendicular to the vertical line through B ; then

if B is higher than A (Fig. 38), $\hat{CAB}$ is the angle of elevation of B as viewed from A

and if B is lower than A (Fig. 39), $\hat{CAB}$ is the angle of depression of B as viewed from A .

Example 1. A ladder 24 ft. long just reaches the top of a wall 12 ft. high ; find the angle at which the ladder is inclined to the horizontal.

Let AB (Fig. 40) represent the ladder, and let CB represent the height of the wall.

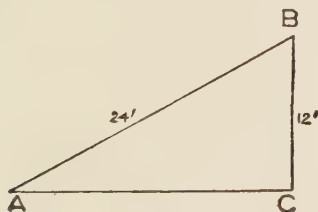


FIG. 40

$$\begin{aligned}\text{Then } \sin A &= \frac{CB}{AB} \\ &= \frac{12}{24} \\ &= \frac{1}{2} \\ \therefore A &= 30^\circ\end{aligned}$$

i.e., the ladder is inclined to the horizontal at angle 30° .

Example 2. From the top of a lighthouse, 150 ft. high, the angle of depression of a ship at sea is observed to be 4° ; find the distance of the ship from the lighthouse.

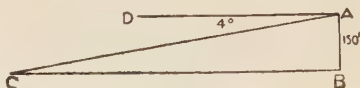


FIG. 41

Let **AB** (Fig. 41) represent the lighthouse, and let **C** represent the ship.

Then, if **AD** is parallel to **BC**, $\hat{DAC} = 4^\circ$, and $\therefore$ also $\hat{ACB} = 4^\circ$.

Hence $\hat{BAC} = 86^\circ$

$$\begin{aligned} \text{and } \therefore BC &= AB \tan 86^\circ \\ &= 150 \text{ ft.} \times 14.30 \\ &= 2145 \text{ ft.} \\ &= 715 \text{ yds.} \end{aligned}$$

$\therefore$ distance of ship from lighthouse is 715 yds.

Example 3. At two points, one vertically over and 40 ft. higher than the other, an object has angles of depression 50° and 17° respectively; find the height of the lower point above the level of the object.

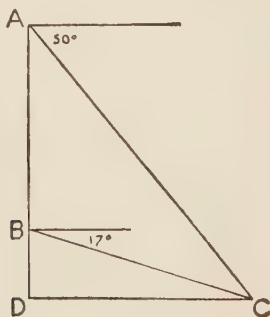


FIG. 42

Let **A** and **B** (Fig. 42) represent the points, and let **C** represent the object.

Let **CD**, perpendicular to **AB**, meet **AB** produced in **D**.

Then $\hat{DAC} = 90^\circ - 50^\circ = 40^\circ$

and $\hat{D}\hat{B}C = 90^\circ - 17^\circ = 73^\circ$.

$\therefore$ letting $BD = x$ ft. we have $DC = BD \tan \hat{D}\hat{B}C = x \tan 73^\circ$

and $DC = AD \tan \hat{D}\hat{A}C = (x + 40) \tan 40^\circ$

$$\therefore x \tan 73^\circ = (x + 40) \tan 40^\circ$$

$$\therefore x (\tan 73^\circ - \tan 40^\circ) = 40 \tan 40^\circ$$

$$\therefore x (3.2709 - 0.8391) = 40 \times 0.8391$$

$$x = \frac{40 \times 0.8391}{2.4318}$$

$$= 13.8$$

$\therefore$ the lower point is 13.8 ft. above the level of the object.

EXERCISES

1. A kite is held by a taut string, 100 ft. long and inclined to the horizontal at angle 30° ; find the height of the kite above the level of the lower end of the string.

2. A ladder of length 32 ft. has its lower end 16 ft. from the foot of a wall, and just reaches the top of the wall. Determine the angle at which the ladder is inclined to the horizontal.

3. A guy, attached to a vertical pole, is 80 ft. long and inclined to the horizontal at angle 60° ; find the height of the point at which the guy is attached.

4. A and B are points, one on each of the parallel banks of a canal 20 ft. broad. If AB is inclined to a bank at angle 30° find the distance between A and B.

5. In a rectangle ABCD the side AB is half the length of a diagonal; find the size of the angle DBC.

6. A, B, C are three points in a horizontal plane, B being east of A, while C is north of B and 30° east of north of A. If A is one mile from B find the distance from A to C.

7. If M is the mid-point of the side AB of a square ABCD determine the size of the angle AMD.

8. A ladder rests against the top of a wall 22 ft. high and has its lower end 8 ft. from the foot of the wall. Determine the angle at which the ladder is inclined to the wall.

9. From a ship at sea the top of a lighthouse is observed to have angle of elevation $2^\circ 30'$. If the ship is one mile from the lighthouse find the height of the latter.

10. A horizontal telephone wire connecting two houses, one on each

side of a street, is inclined at 30° to the direction of the street. Compare the breadth of the street with the length of the wire.

11. A vertical pole stands on a horizontal plane. If the pole is 7 ft. and its shadow 15 ft. long what is the altitude of the sun?

12. What angle does a vertical tower of height 60 ft. subtend at a point which is distant 37 ft. from the tower and which lies in the horizontal plane through the base of the tower?

13. A vertical pole 17 ft. long stands on a horizontal plane; find the length of the pole's shadow, the altitude of the sun being 57° .

14. **A**, **B**, **C** are three points in a horizontal plane, **B** being north of **A**. Determine the bearing of **C** as seen from **A**, if $AB : BC : CA = 5 : 12 : 13$.

15. A man observes a tower to be due east, but after walking one mile due north he finds the tower bears S.S.E. Find the distances of the tower from the two points of observation.

16. **A** and **B**, two points 50 ft. apart, lie on a horizontal line through the base of a vertical tower and are on the same side of the tower. If the angles of elevation of the top of the tower as viewed from **A** and **B** are 50° and 30° respectively find the height of the tower.

17. At a point on the same level as and distant x from the foot of a vertical rod, the angle of elevation of the mid-point of the rod is α . Show that the height of the rod is $2x \tan \alpha$.

18. A tower 180 ft. high has elevation 57° as observed from a point on the same level as the base of the tower; determine the elevation as viewed from a second point on the same level but 50 yds. farther from the tower.

19. From the ends of a straight horizontal road, one mile long, a balloon vertically over the road is observed to have elevations 63° and 27° respectively. Find the height of the balloon above the level of the road.

20. **A** bears 30° east of south of **B**, and is 160 miles from **B**, while **B** is 40 miles east of **C**. Determine the bearing of the line **AC**.

21. In the quadrilateral **ABCD** the angles **A** and **B** are right angles. Find the size of the acute angle between **AB** and **CD**, if the sides **AB**, **BC**, and **DA** are respectively 10, 12.3, and 2.4 ins. long.

22. A vertical rod stands on a horizontal plane, the shadow of the rod being 18 ft. long when the sun has altitude 43° . Determine the length of the shadow when the sun's altitude is 32° .

23. Find, to the nearest degree, the sizes of the angles of a triangle whose sides are respectively 4, 4, and 3 ins. long.

24. A vertical tower subtends angles of 70° and 80° respectively at

two points in the horizontal plane through its base. If one of the points is 20 ft. farther from the tower than the other is, determine, to the nearest foot, the height of the tower and the distances of its top from the two points.

25. In $\triangle ABC$, $C = 90^\circ$ and CD , perpendicular to AB , meets AB in D ; show that $CD = a \cos A = b \cos B$.

26. In $\triangle ABC$, AD , perpendicular to BC , meets BC in D ; show that $AD = b \sin C = c \sin B$, and hence that $\frac{b}{\sin B} = \frac{c}{\sin C}$.

27. A vertical rod stands on a horizontal plane; compare the length of its shadow when the sun's altitude is 59° with that when the sun has altitude 27° .

28. From a lighthouse a ship one mile distant is observed to have angle of depression 3° , while a balloon vertically over the ship is observed to have angle of elevation 48° . Find the height of the balloon above sea-level.

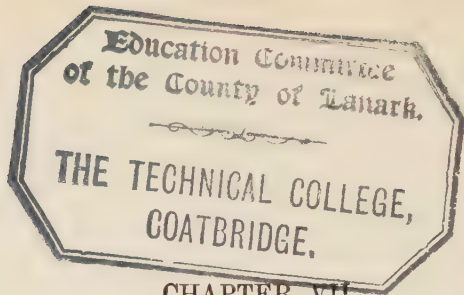
29. Two vertical towers standing on a horizontal plane are 100 yds. apart; at the top of one which is 80 ft. high the other subtends an angle of 30° . Find, to the nearest foot, the height of the other tower.

30. From a ship at sea the angle of elevation of a balloon is observed to be $51^\circ 32'$, while from an airship vertically over the ship and 1000 ft. above it the balloon is observed to have elevation $8^\circ 54'$. Find the height of the balloon.

31. A vertical rod standing on a horizontal plane casts a shadow 23 ft. long when the sun's altitude is 37° . Determine the sun's altitude when the shadow is 43 ft. long.

32. From a ship sailing due north a lighthouse is observed to bear N.E. Later, when the ship is two miles from the lighthouse, the latter bears S.S.E. Find the distance of the lighthouse from the first point of observation.

33. One ship is due east of another and one mile from it; the ships bear N.W. and N.W. by N. respectively, as observed from a coast-guard station. Determine the distances of the ships from the point of observation.



CHAPTER VII

GRAPHS OF CIRCULAR FUNCTIONS

§ 17. THE GRAPH OF AN EQUATION

Consider the equation $y = \sin x^\circ$. All points whose coordinates, x and y , with reference to a chosen pair of axes, satisfy the equation $y = \sin x^\circ$ lie on a curve called 'the graph of the equation $y = \sin x^\circ$ '; the equation is referred to as 'the equation of the graph.'

Note 1. Sometimes the equation is said to be 'represented' by the graph, while the graph is said to be 'given' by the equation.

Note 2. The graphs of some equations consist of a number of curves; see §§ 20, 21.

Note 3. To draw the graph of a given equation we plot a number of points whose coordinates satisfy the given equation, and then draw the curve or curves suggested by the manner in which the points are arranged; see § 18.

§ 18. THE GRAPH OF THE EQUATION $y = \sin x^\circ$

If x is given any value the corresponding value of y can be obtained from the table of sines. A series of such pairs of corresponding values of x and y are entered in the following table, the values of y being given to two decimal places.

x	-30	0	30	60	90	120	150	180	210	240	270	300	330	360	390	420
y	-0.50	0.00	0.50	0.87	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00	-0.87	-0.50	0.00	0.50	0.87

Since the values of x are large numerically compared with those of y , it is convenient to make the unit on the x -axis small compared with that on the y -axis. For this reason, in Fig. 43, 1 in. on the x -axis represents 90, while 1 in. on the y -axis represents unity.

The crosses in Fig. 43 indicate the points whose coordinates are given in the table. The curve which is drawn through these points is therefore the graph of the equation $y = \sin x^\circ$.

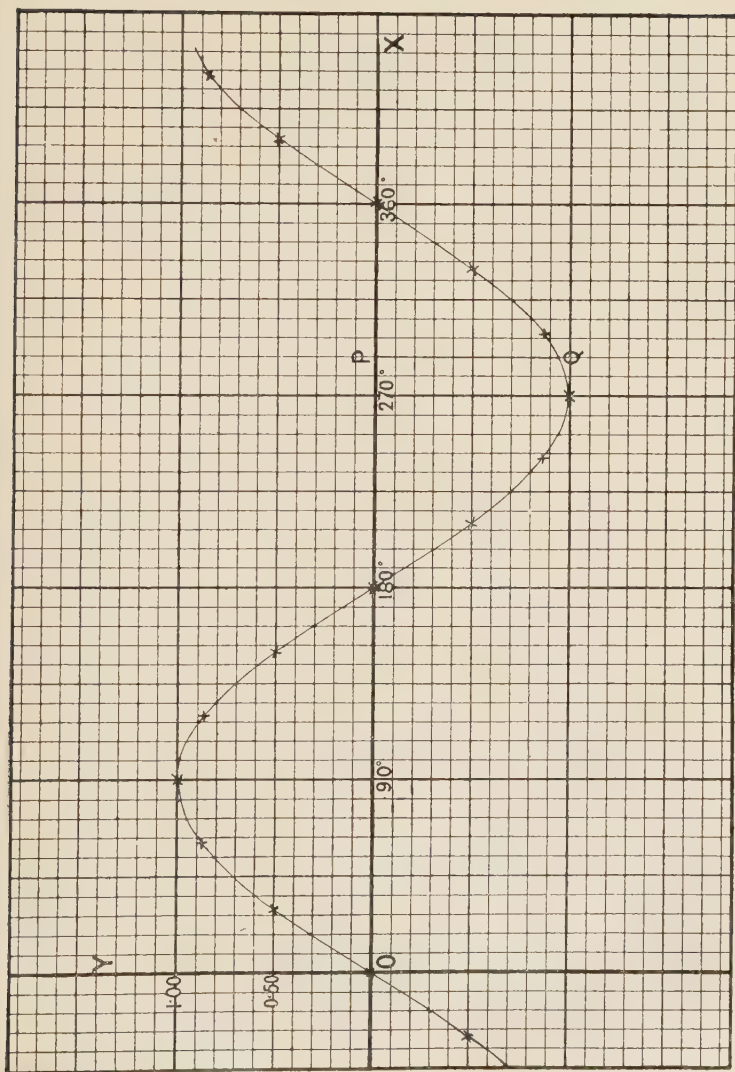


FIG. 43. GRAPH OF $y = \sin x$

Note 1. Since $\sin(360^\circ + x^\circ) = \sin x^\circ$ for all values of x , the graph of the equation $y = \sin x^\circ$ for the range $x = 360$ to $x = 720$ is a curve identically equal to the graph for the range $x = 0$ to $x = 360$, and the complete graph of the equation $y = \sin x^\circ$ is an unending curve consisting of an infinite number of parts each identically equal to the curve for the range $x = 0$ to $x = 360$.

Note 2. From that part of the graph which is shown in Fig. 43 the sines of all angles which lie between -30° and $+420^\circ$ can be determined approximately; *e.g.*, to determine $\sin 288^\circ$, draw the ordinate PQ corresponding to $x = 288$; by measurement $PQ = 0.95$ in., $\therefore \sin 288^\circ = -0.95$.

Note 3.

As x increases from 0 to 90 , $\sin x^\circ$ increases from 0 to 1.
 " " 90 to 180, " decreases " 1 to 0.
 " " 180 to 270, " decreases " 0 to -1 .
 " " 270 to 360, " increases " -1 to 0.

The value of $\sin x^\circ$ goes through a complete series of changes as x varies from 0 to 360; further increase of the value of x merely repeats these changes in the value of $\sin x^\circ$. For this reason $\sin x$ is called a 'periodic' function of x , the 'period' being 360° ; using circular measure, we say that $\sin \theta$ is a periodic function of θ , the period being 2π .

Note 4. As the graph does not change in type according to the unit of angle chosen, it is customary to omit the degree sign in the equation $y = \sin x^\circ$, and to refer to the graph as that of the equation $y = \sin x$, or simply as the graph of $\sin x$.

§ 19. THE GRAPH OF THE EQUATION $y = \cos x$

From the table of cosines we obtain the following pairs of corresponding values of x and y :

x	-30°	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°	390°	420°
y	0.87	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00	-0.87	-0.50	0.00	0.50	0.87	1.00	0.87	0.50

Taking 1 in. on the x -axis to represent 90° , and 1 in. on the y -axis to represent unity, we obtain the graph shown in Fig. 44.

Note 1. The complete graph of $y = \pm \cos x$ is an unending curve consisting of an infinite number of parts each identically equal to that for the range $x = 0^\circ$ to $x = 360^\circ$.

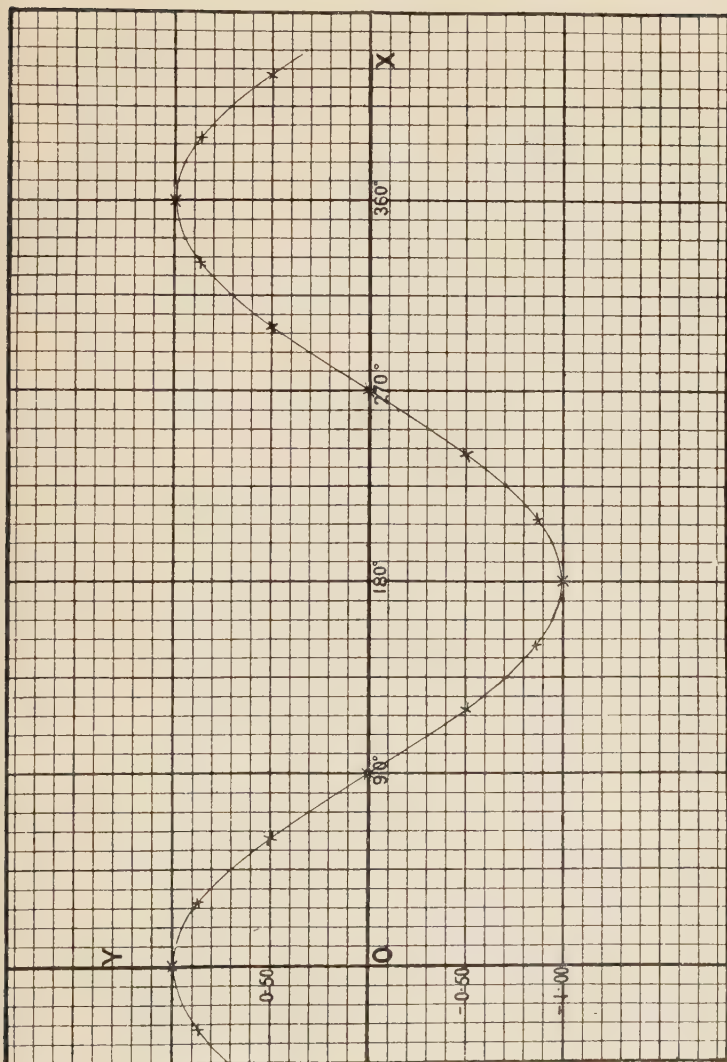
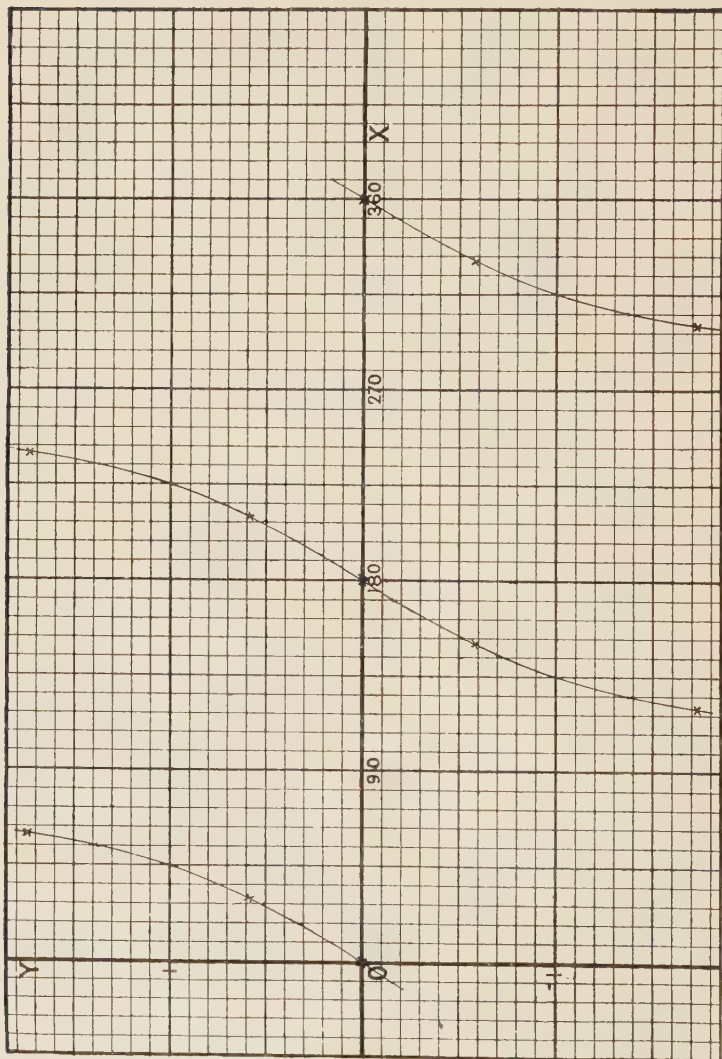


FIG. 44. GRAPH OF $y = \cos x$

FIG. 45. GRAPH OF $y = \tan x$

Note 2.

As x increases from 0° to 90° , $\cos x$ decreases from 1 to 0.
 " " 90° to 180° , " decreases " 0 to -1.
 " " 180° to 270° , " increases " -1 to 0.
 " " 270° to 360° , " increases " 0 to 1.

$\cos x$ is a periodic function of x , and its period is 360° ; using circular measure, $\cos \theta$ is a periodic function of θ , the period being 2π .

 § 20. THE GRAPH OF THE EQUATION $y = \tan x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	0.00	0.58	1.73	∞	-1.73	-0.58	0.00	0.58	1.73	∞	-1.73	-0.58	0.00

Note 1. If x is a little less than 90° , $\tan x$ is very large and positive, and as x approaches the value 90° , $\tan x$ tends to become $+\infty$; if x is a little greater than 90° , $\tan x$ is large numerically but negative, and if x approaches the value 90° , $\tan x$ tends to the value $-\infty$. Thus the graph of $y = \tan x$ must approach the ordinate $x = 90^\circ$ as shown in Fig. 45. In the same way the manner in which the graph approaches the ordinate $x = 270^\circ$ is determined.

Note 2. The complete graph of $y = \tan x$ consists of an infinite number of curves, each identically equal to that for the range $x = -90^\circ$ to $x = +90^\circ$, or $x = 90^\circ$ to $x = 270^\circ$.

Note 3.

As x increases from 0° to 90° , $\tan x$ increases from 0 to ∞ .
 " " 90° to 180° , " " $-\infty$ to 0.
 " " 180° to 270° , " " 0 to ∞ .
 " " 270° to 360° , " " $-\infty$ to 0.

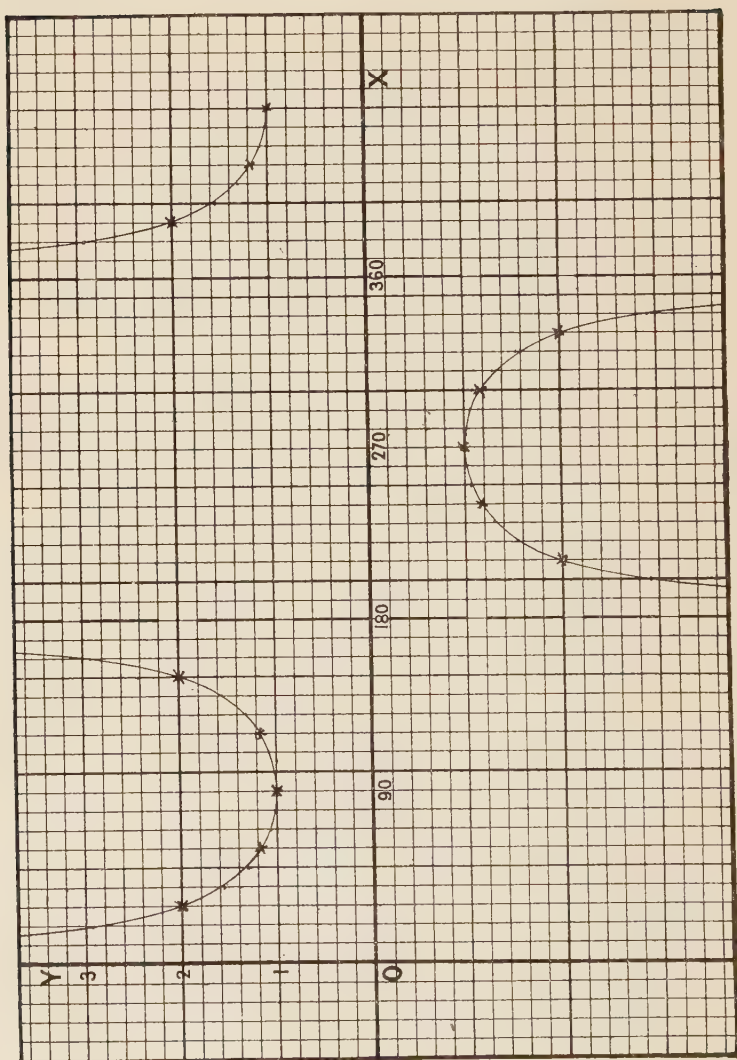
$\tan \theta$ is a periodic function of θ , and has period π .

 § 21. THE GRAPHS OF $\operatorname{cosec} x$, $\sec x$, AND $\cot x$

Note 1. The student should verify the manner in which the graphs of $\operatorname{cosec} x$, $\sec x$, and $\cot x$ (Figs. 46, 47, 48, respectively) approach infinity, and should observe the variations in the values of these functions as x increases from 0° to 360° .

Note 2. The functions $\operatorname{cosec} \theta$, $\sec \theta$, and $\cot \theta$ are periodic functions of θ , and have periods 2π , 2π , and π respectively.

§ 22. The examples on pages 67 and 70 concern the graphs of some equations involving circular functions, and show how such graphs may be utilized.

FIG. 46. GRAPH OF $y = \csc x$

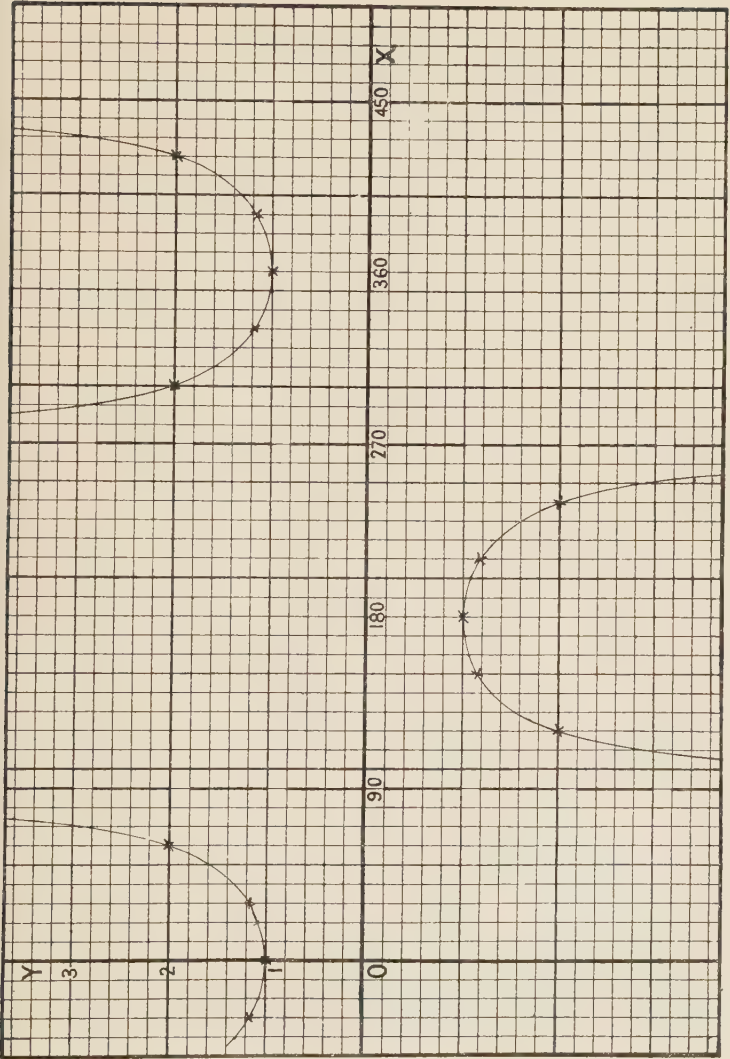


FIG. 47. GRAPH OF $y = \sec x$

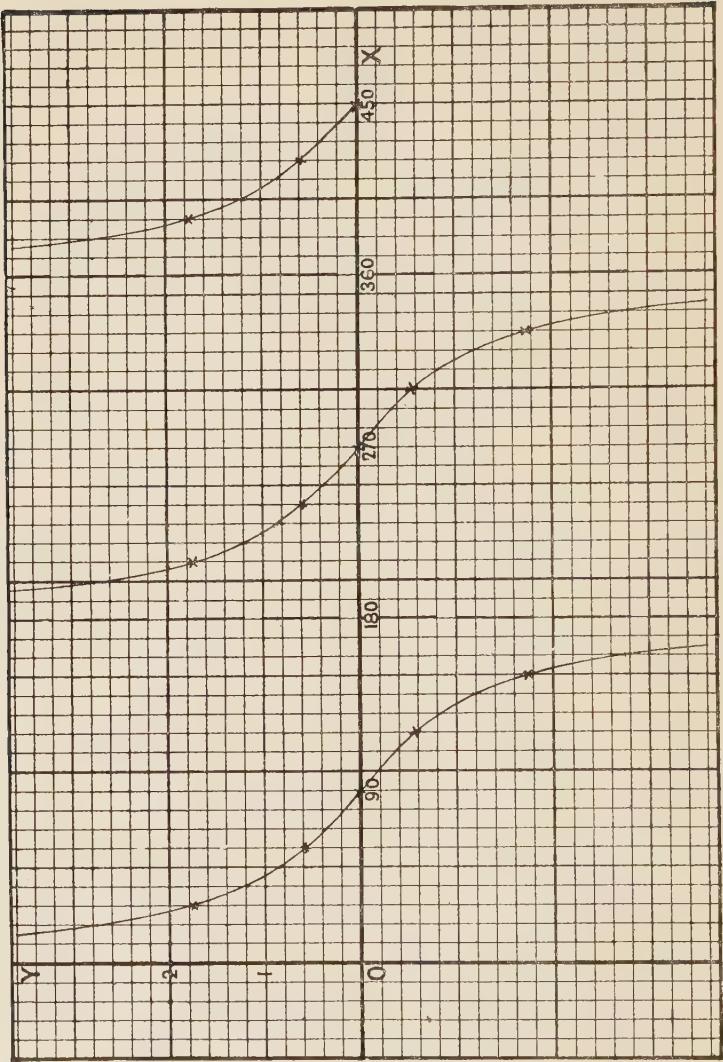


FIG. 48. GRAPH OF $y = \cot x$

Example 1. Draw the graphs of $y = \sin x$ and $y = \cos (x - 30^\circ)$ from $x = 40^\circ$ to $x = 90^\circ$, choosing 1 in. on the x -axis to represent 10° , and 5 in. on the y -axis to represent unity. Hence determine a solution of the equation $\sin x = \cos (x - 30^\circ)$.

x	40°	50°	60°	70°	80°	90°
$\sin x$	0.643	0.766	0.866	0.940	0.985	1.000
$\cos (x - 30^\circ)$	0.985	0.940	0.866	0.766	0.643	0.500

At **A** (Fig. 49), the point in which the curves intersect, x is seen from the figure to have the value 60° . Therefore, when $x = 60^\circ$, $\sin x = \cos (x - 30^\circ)$; i.e. $x = 60^\circ$ is a solution of the equation $\sin x = \cos (x - 30^\circ)$.

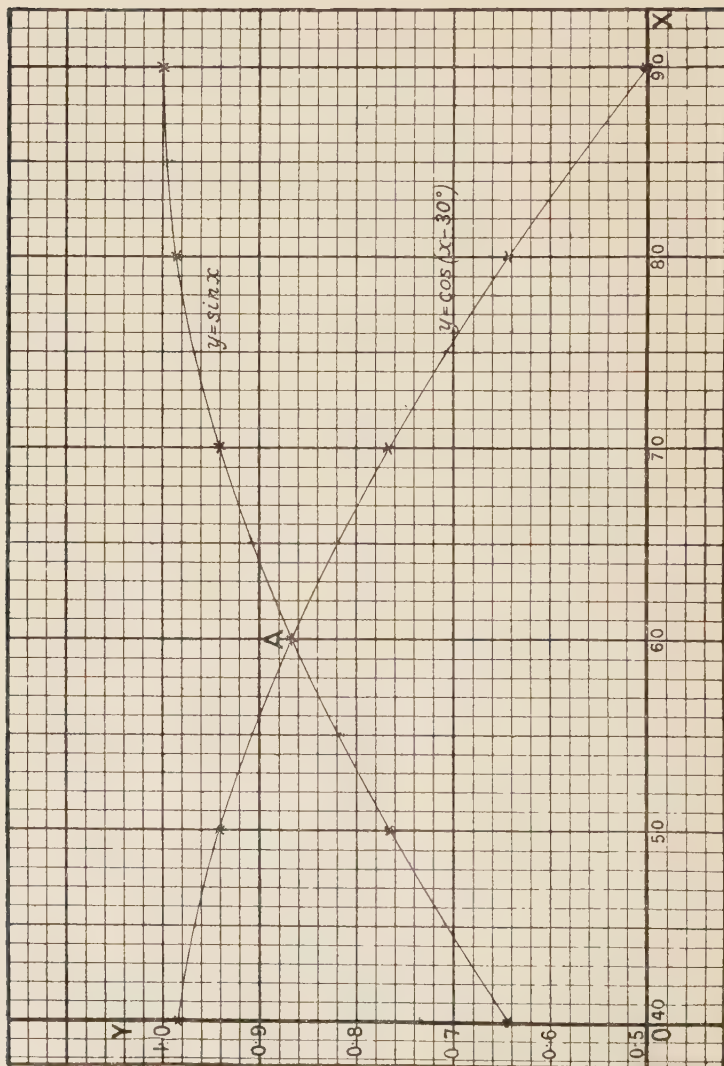
Note 1. Since the range of variation of x is from $x = 40^\circ$ to $x = 90^\circ$ the origin has been chosen as the point on the x -axis at which $x = 40^\circ$. In the same way the origin has been chosen as the point on the y -axis at which $y = 0.5$.

Note 2. Since 5 in. on the y -axis represents unity the values of $\sin x$ and $\cos (x - 30^\circ)$ have been taken to three decimal places: on subtracting 0.500 from these values and multiplying the remainders by 5, the lengths in inches of the different ordinates were obtained correct to two decimal places, an accuracy convenient in plotting points.

Example 2. Taking 1 in. on the x -axis to represent 10° , and 1 in. on the y -axis to represent unity, draw the graphs of $y = \cos x$ and $y = \tan 2x$ for the range $x = 0^\circ$ to $x = 35^\circ$, and hence determine a solution of the equation $\tan 2x = \cos x$.

x	0°	5°	10°	15°	20°	25°	30°	35°
$\cos x$	1.00	1.00	0.98	0.97	0.94	0.91	0.87	0.82
$\tan 2x$	0.00	0.18	0.36	0.58	0.84	1.19	1.73	2.75

At **A** (Fig. 50), the point in which the curves intersect, $x = 21\frac{1}{2}^\circ$. $\therefore x = 21\frac{1}{2}^\circ$ is a solution of the equation $\tan 2x = \cos x$.



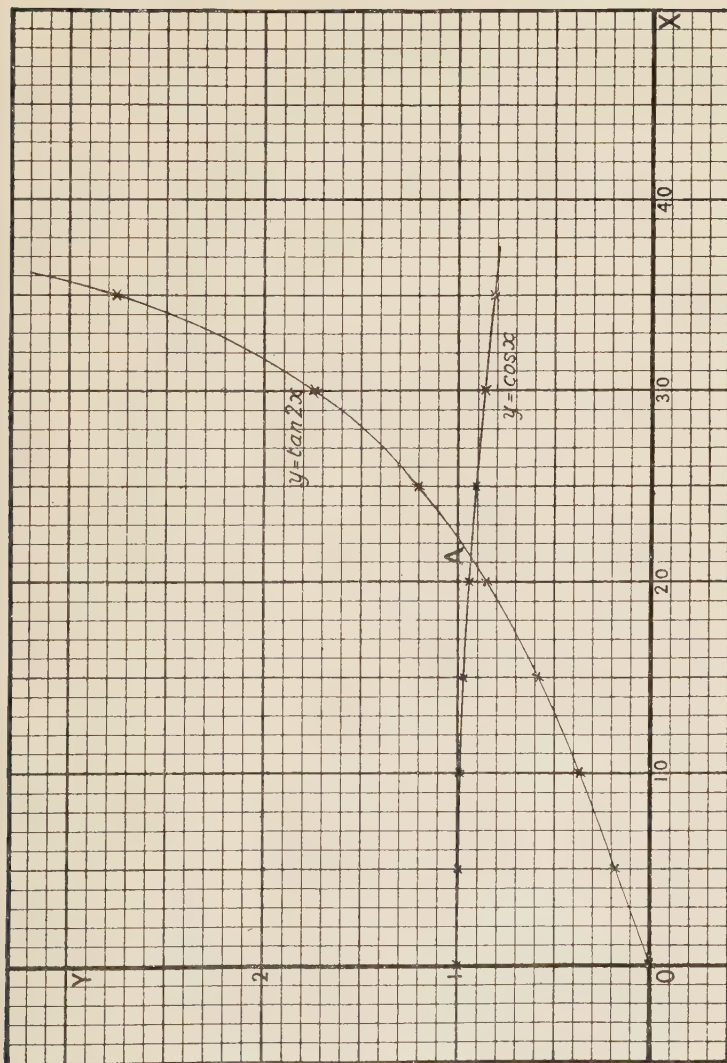


FIG. 50

Example 3. Determine graphically how many roots of the equation $\sin 3x = 2 \tan x$ are greater than 0° but less than 180° .

x	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
$\sin 3x$	0.00	0.50	0.87	1.00	0.87	0.50	0.00	-0.50	-0.87	-1.00
$2 \tan x$	0.00	0.35	0.73	1.15	1.68	2.38	3.46	5.49	11.34	∞

x	100°	110°	120°	130°	140°	150°	160°	170°	180°
$\sin 3x$	-0.87	-0.50	0.00	0.50	0.87	1.00	0.87	0.50	0.00
$2 \tan x$	-11.34	-5.49	-3.46	-2.38	-1.68	-1.15	-0.73	-0.35	0.00

In the preceding table we have entered corresponding values of x , $\sin 3x$, and $2 \tan x$, and by means of these values have drawn (Fig. 51) the graphs of $y = \sin 3x$ and $y = 2 \tan x$. These graphs intersect in only one point lying between the ordinates $x = 0^\circ$ and $x = 180^\circ$, and $\therefore$ the equation $\sin 3x = 2 \tan x$ has only one root which is greater than 0° but less than 180° .

Note. In Fig. 51 we have omitted to plot some points which lie on the graph of $y = 2 \tan x$ and whose coordinates are given in the table. These points, having numerically large ordinates, do not lie on those portions of the $2 \tan x$ curve which are close to the $\sin 3x$ curve, and hence are not required in determining the intersection of these curves.

Example 4. Solve graphically the equation
 $\cot x = 10 (\sin x - \cos x)$,
giving solutions between 10° and 60° .

x	10°	20°	30°	40°	50°	60°
$\cot x$	5.67	2.75	1.73	1.19	0.84	0.58
$10 \sin x$	1.736	3.420	5.000	6.428	7.660	8.660
$10 \cos x$	9.848	9.397	8.660	7.660	6.428	5.000
$10 (\sin x - \cos x)$	-8.11	-5.98	-3.66	-1.23	1.23	3.66

In Fig. 52 the graphs of $y = \cot x$ and $y = 10 (\sin x - \cos x)$ are drawn. At A, the only point in which these graphs intersect, $x = 48\frac{1}{2}^\circ$, and $\therefore x = 48\frac{1}{2}^\circ$ is the only solution of the equation $\cot x = 10 (\sin x - \cos x)$, lying between 10° and 60° .

EXERCISES

1. Draw the graphs of $y = \sin x$ and $y = \cos x$ from $x = 0^\circ$ to $x = 60^\circ$, choosing 1 in. on the x -axis to represent 10° , and 5 in. on the y -axis to represent unity, and hence determine a solution of the equation $\sin x = \cos x$, lying between 0° and 60° .

2. Taking 1 in. on the x -axis to represent 10° , and 4 in. on the y -axis to represent unity, draw the graphs of $y = \sin 2x$ and $y = \tan x$ from $x = 0^\circ$ to $x = 60^\circ$. Hence find that value of x , lying between 0° and 60° , for which $\sin 2x = \tan x$.

3. Choosing 1 in. on the x -axis to represent 10° , and 5 in. on the y -axis to represent unity, draw the graph of $y = \tan^2 x$ from $x = 0^\circ$ to $x = 45^\circ$. Using the same axes, draw the graph of $y = \cos x$, and thus determine a solution of the equation $\tan^2 x = \cos x$, lying between 0° and 45° .

4. Taking 1 in. on the x -axis to represent 20° and 10 in. as unit on the y -axis, draw the graph of $y = \sin x + \cos x$ from $x = 0^\circ$ to $x = 90^\circ$, and hence determine two solutions of the equation $\sin x + \cos x = 1.2$, lying between 0° and 90° .

5. Letting 1 in. on the x -axis represent 10 and on the y -axis 20 units, draw the graphs of $y = x + 20$ and $y = 100 \cos x^\circ$ from $x = 0$ to $x = 60$; hence determine a solution of the equation $x + 20 = 100 \cos x^\circ$.

6. Draw the graphs of $y = \tan(x + 120^\circ)$ and $y = -\sin x$ from $x = 240^\circ$ to $x = 300^\circ$, choosing 1 in. on the x -axis to represent 10° , and 4 in. on the y -axis to represent unity, and thus solve the equation $\tan(x + 120^\circ) + \sin x = 0$ for values of x lying between 240° and 300° .

7. Draw the graphs of $y = \tan x$ and $y = \cos x$ for the range $x = 0^\circ$ to $x = 60^\circ$, and thus solve the equation $\tan x = \cos x$ for values of x lying between 0° and 60° .

8. By drawing the graphs of $y = \operatorname{cosec} x$ and $y = 3 \tan x$ determine to the nearest degree a value of x between 0° and 45° , such that $\operatorname{cosec} x = 3 \tan x$.

9. Draw the graphs of $y = \tan x$ and $y = 2 \sin x$ from $x = 0^\circ$ to $x = 60^\circ$, and hence determine whether there is a value of x , within the range $x = 0^\circ$ to $x = 60^\circ$, which satisfies the equation $\tan x = 2 \sin x$.

10. Draw the graphs of $\cos x$ and $\sin^2 x$ for the range $x = 30^\circ$ to $x = 90^\circ$, and hence determine a value of x , lying between 30° and 90° , for which $\sin^2 x = \cos x$.

11. Draw the graph of $y = 5000 \cos x^\circ$ from $x = 40$ to $x = 80$. Using the same axes, draw the graph of $y = x^2$, and hence solve the equation $x^2 = 5000 \cos x^\circ$ for values of x lying between 40 and 80.

12. Draw the graphs of $y = \cos 3x$ and $y = \sin 2x - 0.37$ from $x = 0^\circ$ to $x = 45^\circ$, and thus solve the equation $\sin 2x - \cos 3x = 0.37$ for values of x lying between 0° and 45° .

13. Determine graphically the two values of x , lying between 0° and 90° , for which $\sin x + \cos x = 1.32$. Verify that and explain why these two values are complementary.

14. Draw the graphs of $y = \sec x$ and $y = 2 \cot x$ for the range $x = 40^\circ$ to $x = 60^\circ$, and hence solve the equation $\sin x = 2 \cos^2 x$ for values of x lying between 40° and 60° .

15. Solve graphically, for values of x lying between 45° and 90° , the equation $\tan x - \sin x = 10 \cos x$.

16. Solve graphically the equation $\sin 2x = \cos 2x$, giving solution between 0° and 45° .

17. Solve graphically the equation $\tan 2x = 8 \cos^2 x$, giving solutions between 0° and 45° .

18. Determine graphically a value of x , lying between 45° and 75° , for which $2 \tan^2 x = \tan x + 3$.

19. Find graphically an acute angle θ which satisfies the equation $\sin^2 \theta = 3 \sin \theta - 1$.

20. Solve graphically the equation $\sin 2(x - 30^\circ) = \cos x - 1$ for the range $x = 0^\circ$ to $x = 45^\circ$.

21. Determine graphically an obtuse angle ϕ such that

$$\sin \left(\frac{\phi}{2} + 45^\circ \right) = \tan \left(\frac{\phi}{2} - 30^\circ \right).$$

22. Find graphically a value of x lying between -45 and zero, such that $x^2 = 2000 \tan (x^\circ + 30^\circ)$.

23. Show that there are two values of θ , lying between $\frac{\pi}{2}$ and 3π , which satisfy the equation $\sin \frac{\theta}{4} = \tan \theta$.

24. Determine graphically the angle θ radians, lying between 0 and π , such that $\theta = 2 \sin \theta$.

25. Draw the graphs of $y = \cos \theta$ and $y^2 = \theta - \frac{\pi}{8}$, θ radians being an acute angle, and so determine a solution of the equation $6(\theta - \cos^2 \theta) = \pi$.

The following exercises may be attempted at this stage by students familiar with logarithms; others may delay their attempt until they have completed Chapter XI.

26. Determine a value of x , lying between 10° and 60° , for which $\sin (x - 60^\circ) = \log \cos x$.

27. Solve graphically, giving solutions between 10° and 70° , the equation $\cos x = 1 + \log \tan x$.

28. Find the value of x , lying between 10° and 60° , which satisfies the equation $\sin x = 10^{\sin(10-x)}$.

29. For what value of x greater than 10 but less than 60 is $\log x^2 = 2 \tan x^\circ + 1$?

30. Solve the equation $x = 10^{1+\cos x^\circ}$ for values of x lying between 10 and 60.

CHAPTER VIII

CIRCULAR FUNCTIONS OF COMPOUND ANGLES

§ 23. CIRCULAR FUNCTIONS OF THE SUM AND DIFFERENCE OF TWO ANGLES

Theorem I. To prove (i) $\sin (A+B) = \sin A \cos B + \cos A \sin B$.
(ii) $\cos (A+B) = \cos A \cos B - \sin A \sin B$.

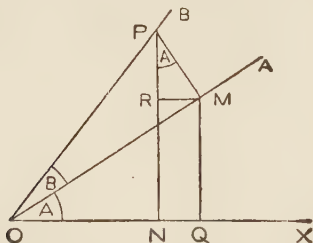


FIG. 53

Let the acute angle $\angle XO A = \hat{A}$ (Fig. 53), and let the acute angle $\angle AO B = \hat{B}$

Then $\angle XO B = \text{angle } (A+B)$; let this angle also be acute.

From P, any point on OB, draw PM, PN perpendicular to OA, OX respectively; draw MQ, MR perpendicular to OX, NP respectively.

Then, since $\angle ONP = \angle OMP$, O, P, M, and N are concyclic

$$\therefore \hat{NPM} = \hat{NOM}, \text{ i.e. } = \hat{A}$$

$$\text{and (i) } \sin (A+B) = \frac{NP}{OP}$$

$$= \frac{NR + RP}{OP}$$

$$= \frac{QM + RP}{OP}$$

$$= \frac{QM}{OP} + \frac{RP}{OP}$$

$$= \frac{QM}{OM} \cdot \frac{OM}{OP} + \frac{RP}{MP} \cdot \frac{MP}{OP}$$

$$= \sin A \cos B + \cos A \sin B.$$

$$\begin{aligned}
 \text{(ii) } \cos(A+B) &= \frac{ON}{OP} \\
 &= \frac{OQ - NQ}{OP} \\
 &= \frac{OQ - RM}{OP} \\
 &= \frac{OQ}{OP} - \frac{RM}{OP} \\
 &= \frac{OQ}{OM} \cdot \frac{OM}{OP} - \frac{RM}{MP} \cdot \frac{MP}{OP} \\
 &= \cos A \cos B - \sin A \sin B.
 \end{aligned}$$

Theorem II. To prove (i) $\sin(A-B) = \sin A \cos B - \cos A \sin B$.
 (ii) $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

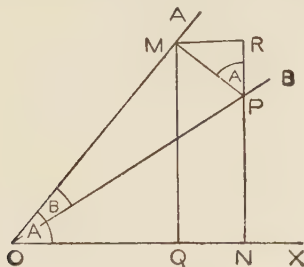


FIG. 54

Let the acute angle $XOA = \hat{A}$ (Fig. 54), and let the acute angle $AOB = \hat{B}$.

Then $X\hat{O}B = \text{angle } (A-B)$; since $\hat{A}$ is taken greater than $\hat{B}$, angle $(A-B)$ is also acute.

From P, any point on OB, draw PM, PN perpendicular to OA, OX respectively; draw MQ, MR perpendicular to OX, NP respectively.

Then, since $\hat{ONP} = \hat{OMP} = 90^\circ$, O, N, P, and M are concyclic

$$\therefore \hat{RPM} = \hat{NOM}, \text{ i.e. } = \hat{A},$$

$$\begin{aligned}
 \text{and (i) } \sin(A-B) &= \frac{NP}{OP} \\
 &= \frac{QM - PR}{OP} \\
 &= \frac{QM}{OM} \cdot \frac{OM}{OP} - \frac{PR}{MP} \cdot \frac{MP}{OP} \\
 &= \sin A \cos B - \cos A \sin B.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } \cos (A - B) &= \frac{ON}{OP} \\
 &= \frac{OQ + MR}{OP} \\
 &= \frac{OQ}{OM} \cdot \frac{OM}{OP} + \frac{MR}{MP} \cdot \frac{MP}{OP} \\
 &= \cos A \cos B + \sin A \sin B.
 \end{aligned}$$

Note 1. The preceding four relations, commonly called *addition formulæ*, are true for all sizes of angles A and B , positive or negative. We shall make use of this fact, but shall not prove it here; the proof will be found in the Appendix.

Note 2. As an example of the proof of these relations when the angles are not all acute, we give the following proof of the relation $\cos (A + B) = \cos A \cos B - \sin A \sin B$, when A and B are acute but angle $(A + B)$ is obtuse.

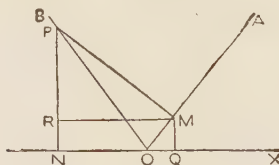


FIG. 55

The construction is the same as in Theorem I.

$\widehat{RPM}$ (Fig. 55) is equal to $\widehat{A}$, and

$$\begin{aligned}
 \cos (A + B) &= \frac{ON}{OP} \\
 &= \frac{OQ - NQ}{OP} \\
 &= \frac{OQ}{OM} \cdot \frac{OM}{OP} + \frac{RM}{MP} \cdot \frac{MP}{OP} \\
 &= \cos A \cos B - \sin A \sin B.
 \end{aligned}$$

Note 3. The identities of Theorems I and II, viz.,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$

must be memorized. The student may observe that

(i) associated with the sine of a sum or difference we have terms containing both a sine and a cosine, while associated with the

cosine of a sum or difference we have terms containing sines only or cosines only.

(ii) in the case of the sine of a sum or difference we have like signs, while in the case of the cosine of a sum or difference we have unlike signs in the identities.

Theorem III. To prove (i) $\tan (A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

$$(ii) \tan (A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

$$\begin{aligned} (i) \tan (A+B) &= \frac{\sin (A+B)}{\cos (A+B)} \\ &= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \end{aligned}$$

on dividing numerator and denominator by $\cos A \cos B$.

$$\begin{aligned} (ii) \tan (A-B) &= \frac{\sin (A-B)}{\cos (A-B)} \\ &= \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B} \\ &= \frac{\tan A - \tan B}{1 + \tan A \tan B} \end{aligned}$$

on dividing numerator and denominator by $\cos A \cos B$.

Note. Since Theorems I and II are true for all values of A and B , Theorem III is also true for all values of A and B .

Example 1. Using the formulæ of § 23, show that

$$\begin{aligned} \operatorname{cosec} (270^\circ - A) &= -\sec A. \\ \operatorname{cosec} (270^\circ - A) &= \frac{1}{\sin (270^\circ - A)} \\ &= \frac{1}{\sin 270^\circ \cos A - \cos 270^\circ \sin A} \\ &= -\frac{1}{\cos A} \quad (\sin 270^\circ = -1 \text{ and } \cos 270^\circ = 0) \\ &= -\sec A. \end{aligned}$$

Example 2. Show that in any triangle ABC

$$\sin 2A + \sin 2B \cos 2C + \cos 2B \sin 2C = 0.$$

$$\begin{aligned} \sin 2A + \sin 2B \cos 2C + \cos 2B \sin 2C &= \sin 2A + \sin (2B + 2C) \\ &= \sin 2A + \sin (360^\circ - 2A) \\ &= \sin 2A - \sin 2A \\ &= 0. \end{aligned}$$

Example 3. Prove that

$$\begin{aligned}
 \sin (A+B+C) &= \sin A \cos B \cos C + \sin B \cos C \cos A \\
 &\quad + \sin C \cos A \cos B - \sin A \sin B \sin C \\
 \sin (A+B+C) &= \sin (\overline{A+B}+C) \\
 &= \sin \overline{A+B} \cos C + \cos \overline{A+B} \sin C \\
 &= (\sin A \cos B + \cos A \sin B) \cos C \\
 &\quad + (\cos A \cos B - \sin A \sin B) \sin C \\
 &= \sin A \cos B \cos C + \sin B \cos C \cos A \\
 &\quad + \sin C \cos A \cos B - \sin A \sin B \sin C.
 \end{aligned}$$

Note. In Example 3 the sine of $(A+B+C)$ is first expressed in terms of the circular functions of the angles $\overline{A+B}$ and C ; the circular functions of the angle $\overline{A+B}$ are then expressed in terms of those of the angles A and B . This process is a useful one and should be remembered.

EXERCISES

1. Using the formulæ of § 23, prove that

$$\begin{array}{ll}
 \text{(i) } \sin (90^\circ - A) = \cos A. & \text{(vi) } \sec (360^\circ - A) = \sec A. \\
 \text{(ii) } \cos (90^\circ - A) = \sin A. & \text{(vii) } \operatorname{cosec} (90^\circ + A) = \sec A. \\
 \text{(iii) } \sin (90^\circ + A) = \cos A. & \text{(viii) } \cos (360^\circ - A) = \cos A. \\
 \text{(iv) } \sin (270^\circ - A) = -\cos A. & \text{(ix) } \tan (270^\circ - A) = \cot A. \\
 \text{(v) } \cos (270^\circ + A) = \sin A. & \text{(x) } \cot (180^\circ + A) = \cot A.
 \end{array}$$

2. Prove the identities

$$\begin{aligned}
 \text{(i) } \sin (A + 45^\circ) &= \frac{1}{\sqrt{2}} (\sin A + \cos A). \\
 \text{(ii) } \sin (A - 45^\circ) &= \frac{1}{\sqrt{2}} (\sin A - \cos A). \\
 \text{(iii) } 2 \sin (A + 30^\circ) &= \sqrt{3} \sin A + \cos A. \\
 \text{(iv) } 2 \cos \left(\theta + \frac{\pi}{3} \right) &= \cos \theta - \sqrt{3} \sin \theta. \\
 \text{(v) } \sin \left(\theta + \frac{7\pi}{4} \right) &= \frac{1}{\sqrt{2}} (\sin \theta - \cos \theta). \\
 \text{(vi) } 2 \cos \left(\theta + \frac{4\pi}{3} \right) &= \sqrt{3} \sin \theta - \cos \theta. \\
 \text{(vii) } \tan (A + 45^\circ) &= \frac{1 + \tan A}{1 - \tan A}. \\
 \text{(viii) } \cot \left(\theta - \frac{5\pi}{4} \right) &= \frac{\tan \theta + 1}{\tan \theta - 1}.
 \end{aligned}$$

3. By expressing 15° as $(45^\circ - 30^\circ)$ and 75° as $(45^\circ + 30^\circ)$, show that

$$(i) \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}, \quad (iv) \sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

$$(ii) \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}, \quad (v) \cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$(iii) \tan 15^\circ = 2 - \sqrt{3}, \quad (vi) \tan 75^\circ = 2 + \sqrt{3}.$$

4. Show that in any triangle ABC

$$(i) \sin A = \sin B \cos C + \cos B \sin C.$$

$$(ii) \cos A = \sin B \sin C - \cos B \cos C.$$

$$(iii) \sin \frac{A}{2} = \cos \frac{B}{2} \cos \frac{C}{2} - \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$(iv) \cos \frac{A}{2} = \sin \frac{B}{2} \cos \frac{C}{2} + \cos \frac{B}{2} \sin \frac{C}{2}.$$

$$(v) \tan \frac{A}{2} = \frac{1 - \tan \frac{B}{2} \tan \frac{C}{2}}{\tan \frac{B}{2} + \tan \frac{C}{2}}.$$

$$(vi) \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Prove that

$$5. \tan A + \tan B = \sin(A+B)/\cos A \cos B.$$

$$6. \tan A - \tan B = \sin(A-B)/\cos A \cos B.$$

$$7. \cot A + \cot B = \sin(A+B)/\sin A \sin B.$$

$$8. \cot B - \cot A = \sin(A-B)/\sin A \sin B.$$

$$9. \tan \alpha + \cot \beta = \cos(\alpha - \beta)/\cos \alpha \sin \beta.$$

$$10. \cot \beta - \tan \alpha = \cos(\alpha + \beta)/\cos \alpha \sin \beta.$$

$$11. \frac{\sin(\theta - \phi)}{\cos(\theta + \phi)} = \frac{\tan \theta - \tan \phi}{1 - \tan \theta \tan \phi} = \frac{\cot \phi - \cot \theta}{\cot \phi \cot \theta - 1}.$$

$$12. \frac{\sin(\theta + \phi)}{\cos(\theta - \phi)} = \frac{\tan \theta + \tan \phi}{1 + \tan \theta \tan \phi} = \frac{\cot \theta + \cot \phi}{1 + \cot \theta \cot \phi}.$$

$$13. \sin(A+B) \sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A.$$

$$14. \cos(A+B) \cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A.$$

$$15. \sin(A+B) \cos(A-B) = \sin A \cos A + \sin B \cos B.$$

$$16. \cos(A+B) \sin(A-B) = \sin A \cos A - \sin B \cos B.$$

$$17. \sin(\theta + \phi) \cos \phi - \cos(\theta + \phi) \sin \phi = \sin \theta.$$

$$18. \sin\left(\frac{\pi}{4} - \theta\right) \cos\left(\frac{\pi}{4} + \theta\right) + \cos\left(\frac{\pi}{4} - \theta\right) \sin\left(\frac{\pi}{4} + \theta\right) = 1.$$

$$19. \sin(\alpha - \beta) = \cos\left(\frac{\pi}{2} - \alpha\right) \cos \beta - \sin\left(\frac{\pi}{2} - \alpha\right) \sin \beta.$$

$$20. \tan a = \frac{\tan 2a - \tan a}{1 + \tan 2a \tan a}.$$

$$21. \sin(x+y) + \sin(x-y) = 2 \sin x \cos y.$$

$$22. \cos(x-y) - \cos(x+y) = 2 \sin x \sin y.$$

$$23. \tan y + \tan(x-y) = \tan x \{1 - \tan y \tan(x-y)\}.$$

$$24. \tan(A-B) = -\tan\{(B-C) + (C-A)\} \text{ and hence}$$

$$\tan(A-B) + \tan(B-C) + \tan(C-A)$$

$$= \tan(A-B) \tan(B-C) \tan(C-A).$$

$$25. \cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C$$

$$- \cos B \sin C \sin A - \cos C \sin A \sin B.$$

$$26. \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan C \tan A - \tan A \tan B}.$$

$$27. \cot(A+B+C) = \frac{\cot A \cot B \cot C - \cot A - \cot B - \cot C}{\cot B \cot C + \cot C \cot A + \cot A \cot B - 1}.$$

28. In triangle ABC , AD is perpendicular to BC ; if AD , BD , DC are respectively 2 in., 4 in., 1 in. long, show that $\tan \hat{BAC} = \infty$ and that $\hat{BAC} = 90^\circ$.

29. Using Exercise 26, show that in any triangle ABC

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

30. Using Exercise 27, show that, if $A+B+C=90^\circ$,

$$\cot A + \cot B + \cot C = \cot A \cot B \cot C.$$

31. Show that, if $A+B+C=180^\circ$,

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

32. Show that, if $A+B+C=90^\circ$,

$$\tan B \tan C + \tan C \tan A + \tan A \tan B = 1.$$

33. Show that, if $\alpha + \beta = \frac{\pi}{4}$, $\cot \alpha - \cot \alpha \tan \beta - \tan \beta = 1$.

§ 24. CIRCULAR FUNCTIONS OF MULTIPLE ANGLES

To prove (i) $\sin 2A = 2 \sin A \cos A$.

$$(ii) \cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A.$$

$$(iii) \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

$$(i) \sin 2A = \sin(A+A)$$

$$= \sin A \cos A + \cos A \sin A \text{ (by Th. I, § 23)}$$

$$= 2 \sin A \cos A.$$

$$(ii) \cos 2A = \cos(A+A)$$

$$= \cos A \cos A - \sin A \sin A \text{ (by Th. I, § 23)}$$

$$= \cos^2 A - \sin^2 A$$

$$\text{But } \cos^2 A - \sin^2 A = \cos^2 A - (1 - \cos^2 A) = 2 \cos^2 A - 1$$

$$\text{and } \cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A = 1 - 2 \sin^2 A$$

$$\therefore \cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A.$$

$$(iii) \tan 2A = \tan (A + A)$$

$$= \frac{\tan A + \tan A}{1 - \tan A \tan A} \quad (\text{by Th. III, § 23})$$

$$= \frac{2 \tan A}{1 - \tan^2 A}.$$

Note 1. Since the relations of Theorems I and III, § 23, are true for all values of A and B , the relations (i) (ii) (iii) of § 24 are true for all values of A ; *e.g.*, letting $A = \frac{\theta}{2}$, we have by (i) $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$.

Note 2. From (ii) we have

$$\cos^2 A = \frac{1 + \cos 2A}{2}.$$

$$\sin^2 A = \frac{1 - \cos 2A}{2}.$$

These relations are often useful and should be remembered.

§ 25. CIRCULAR FUNCTIONS OF MULTIPLE ANGLES—*continued*

When the angle $2A$ is acute the formulæ of § 24 admit of the following simple geometrical proof :

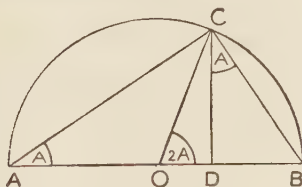


FIG. 56

Let O (Fig. 56) be the mid-point of the line AB ; on AB as diameter construct a semicircle and let $\hat{BOC} = 2A$. Join AC , CB and draw CD perpendicular to AB .

Then $\hat{BAC} = \frac{1}{2}\hat{BOC} = A$; also $\hat{DCB} = 90^\circ - \hat{ACD} = \hat{DAC} = A$

$$\begin{aligned} \text{and (i) } \sin 2A &= \frac{DC}{OC} \\ &= \frac{2DC}{AB} \\ &= 2 \frac{DC}{AC} \cdot \frac{AC}{AB} \\ &= 2 \sin A \cos A. \end{aligned}$$

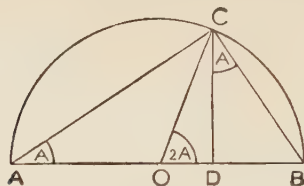


FIG. 56

$$\begin{aligned} \text{(ii) } \cos 2A &= \frac{OD}{OC} \\ &= \frac{2OD}{2OC} \\ &= \frac{AD - DB}{AB} \end{aligned}$$

$$\begin{aligned} &= \frac{AD}{AC} \cdot \frac{AC}{AB} - \frac{DB}{CB} \cdot \frac{CB}{AB} \\ &= \cos^2 A - \sin^2 A. \end{aligned}$$

$$\begin{aligned} \cos 2A &= \frac{OD}{OC} \\ &= \frac{AD - AO}{AO} \end{aligned}$$

$$\begin{aligned} &= \frac{AD}{AO} - 1 \\ &= 2 \frac{AD}{AB} - 1 \\ &= 2 \frac{AD}{AC} \cdot \frac{AC}{AB} - 1 \\ &= 2 \cos^2 A - 1. \end{aligned}$$

$$\begin{aligned} \cos 2A &= \frac{OD}{OC} \\ &= \frac{OB - DB}{OB} \end{aligned}$$

$$\begin{aligned} &= 1 - 2 \frac{DB}{AB} \\ &= 1 - 2 \frac{DB}{CB} \cdot \frac{CB}{AB} \\ &= 1 - 2 \sin^2 A. \end{aligned}$$

$$\begin{aligned} \text{(iii) } \tan 2A &= \frac{DC}{OD} \\ &= \frac{2DC}{AD - DB} \end{aligned}$$

$$\begin{aligned} &= \frac{2 \frac{DC}{AD}}{1 - \frac{DB}{AD}} \end{aligned}$$

$$\begin{aligned}
 &= \frac{2 \frac{DC}{AD}}{1 - \frac{DB}{DC} \cdot \frac{DC}{AD}} \\
 &= \frac{2 \tan A}{1 - \tan^2 A}.
 \end{aligned}$$

§ 26. CIRCULAR FUNCTIONS OF MULTIPLE ANGLES—*continued*

To prove (i) $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$.

(ii) $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$.

$$\begin{aligned}
 \text{(i) } \sin 2A &= 2 \sin A \cos A \\
 &= \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A} \\
 &= \frac{2 \tan A}{1 + \tan^2 A} \quad (\text{on dividing numerator and denominator by } \cos^2 A).
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } \cos 2A &= \cos^2 A - \sin^2 A \\
 &= \frac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} \\
 &= \frac{1 - \tan^2 A}{1 + \tan^2 A} \quad (\text{on dividing numerator and denominator by } \cos^2 A).
 \end{aligned}$$

Note. Since the formulæ of § 24 are true for all values of A , those of § 26 also are true for all values of A .

§ 27. CIRCULAR FUNCTIONS OF MULTIPLE ANGLES—*continued*

To prove (i) $\sin 3A = 3 \sin A - 4 \sin^3 A$.

(ii) $\cos 3A = 4 \cos^3 A - 3 \cos A$.

(iii) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.

$$\begin{aligned}
 \text{(i) } \sin 3A &= \sin (A + 2A) \\
 &= \sin A \cos 2A + \cos A \sin 2A \\
 &= \sin A (1 - 2 \sin^2 A) + \cos A \cdot 2 \sin A \cos A \\
 &= \sin A - 2 \sin^3 A + 2 \sin A (1 - \sin^2 A) \\
 &= 3 \sin A - 4 \sin^3 A.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } \cos 3A &= \cos (A + 2A) \\
 &= \cos A \cos 2A - \sin A \sin 2A \\
 &= \cos A (2 \cos^2 A - 1) - \sin A \cdot 2 \sin A \cos A \\
 &= 2 \cos^3 A - \cos A - 2 \cos A (1 - \cos^2 A) \\
 &= 4 \cos^3 A - 3 \cos A.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii) } \tan 3A &= \tan (A + 2A) \\
 &= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A} \\
 &= \frac{\tan A + \frac{2 \tan A}{1 - \tan^2 A}}{1 - \tan A \frac{2 \tan A}{1 - \tan^2 A}} \\
 &= \frac{\tan A - \tan^3 A + 2 \tan A}{1 - \tan^2 A - 2 \tan^2 A} \\
 &= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.
 \end{aligned}$$

Note 1. The circular functions of $3A$ are first expressed in terms of those of A and $2A$; the circular functions of $2A$ are then expressed in terms of those of A . (Compare § 23, Example 3.)

Note 2. These formulæ are true for all values of A , since the formulæ of §§ 23 and 24 are true for all values of A .

Example 1. Prove that

$$\begin{aligned}
 \cos^4 \theta - \sin^4 \theta &= \cos 2\theta, \\
 \cos^4 \theta - \sin^4 \theta &= (\cos^2 \theta + \sin^2 \theta)(\cos^2 \theta - \sin^2 \theta) \\
 &= 1 \times \cos 2\theta, \\
 &= \cos 2\theta.
 \end{aligned}$$

Example 2. Prove that

$$\begin{aligned}
 \frac{\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}} &= \sec \alpha + \tan \alpha, \\
 \frac{\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}} &= \frac{\left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}\right)\left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}\right)}{\left(\cos \frac{\alpha}{2} - \sin \frac{\alpha}{2}\right)\left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2}\right)} \\
 &= \frac{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2} + 2 \cos \frac{\alpha}{2} \sin \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}} \\
 &= \frac{1 + \sin \alpha}{\cos \alpha} \\
 &= \frac{1}{\cos \alpha} + \frac{\sin \alpha}{\cos \alpha} \\
 &= \sec \alpha + \tan \alpha.
 \end{aligned}$$

Example 3. Prove that

$$\cos 3A + \sin 3A = (\cos A - \sin A)(1 + 2 \sin 2A).$$

$$\begin{aligned}\cos 3A + \sin 3A &= 4 \cos^3 A - 3 \cos A + 3 \sin A - 4 \sin^3 A \\ &= 4 (\cos^3 A - \sin^3 A) - 3 (\cos A - \sin A) \\ &= (\cos A - \sin A) \{4(\cos^2 A + \cos A \sin A + \sin^2 A) - 3\} \\ &= (\cos A - \sin A) \{4(1 + \cos A \sin A) - 3\} \\ &= (\cos A - \sin A)(1 + 4 \cos A \sin A) \\ &= (\cos A - \sin A)(1 + 2 \sin 2A).\end{aligned}$$

Example 4. Prove that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$.

$$\begin{aligned}\sin 36^\circ &= \cos (90^\circ - 36^\circ) \\ &= \cos 54^\circ\end{aligned}$$

$$\therefore 2 \sin 18^\circ \cos 18^\circ = 4 \cos^3 18^\circ - 3 \cos 18^\circ$$

$$\begin{aligned}\therefore 2 \sin 18^\circ &= 4 \cos^2 18^\circ - 3 \\ &= 4(1 - \sin^2 18^\circ) - 3\end{aligned}$$

$$\therefore 4 \sin^2 18^\circ + 2 \sin 18^\circ - 1 = 0$$

$$\text{whence } \sin 18^\circ = \frac{+\sqrt{5}-1}{4} \text{ or } \frac{-\sqrt{5}-1}{4}.$$

The second of these expressions is inadmissible, since $\sin 18^\circ$ is positive

$$\therefore \sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

Example 5. If $x = a \sin \theta$, $y = b \sin 3\theta$, show that $a^3 y = bx(3a^2 - 4x^2)$.

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

$$\therefore \frac{y}{b} = 3 \frac{x}{a} - 4 \left(\frac{x}{a}\right)^3$$

$$= \frac{x}{a^3} (3a^2 - 4x^2)$$

$$\therefore a^3 y = bx(3a^2 - 4x^2).$$

EXERCISES

Prove that

1. $\sin 2\theta = 2 \sin \theta \sin \left(\frac{\pi}{2} - \theta\right).$

2. $\cos 2\theta = \cos^2 \theta - \cos^2 \left(\frac{\pi}{2} - \theta\right).$

3. $(\sin A + \cos A)^2 = 1 + \sin 2A.$

4. $(\sin A - \cos A)^2 = 1 - \sin 2A.$

5. $(\cos \theta + \sin \theta)(\cos \theta - \sin \theta) = \cos 2\theta.$

6. $\cos x - \sin 2x \sin x = \cos x \cos 2x.$
7. $\cos 2A + \cos A = (2 \cos A - 1)(1 + \cos A).$
8. $2 \sin \frac{4\pi}{5} \cos \frac{\pi}{5} = \sin \frac{2\pi}{5}.$
9. $\cos \frac{6\pi}{7} = \cos^2 \frac{4\pi}{7} - \sin^2 \frac{3\pi}{7}.$
10. $\cos \frac{4\pi}{9} = 2 \cos^2 \frac{7\pi}{9} - 1 = 1 - 2 \sin^2 \frac{11\pi}{9}.$
11. $\tan A = \frac{\sin 2A}{1 + \cos 2A}.$
12. $\sec a + \operatorname{cosec} a = \frac{2 (\sin a + \cos a)}{\sin 2a}.$
13. $\sec^2 a + \operatorname{cosec}^2 a = 4 \operatorname{cosec}^2 2a.$
14. $\cot A - \tan A = 2 \cot 2A.$
15. $\cot A + \tan A = 2 \operatorname{cosec} 2A.$
16. $2 \operatorname{cosec} 2x = \operatorname{cosec} x \sec x.$
17. $\cos 2A = \frac{\cot A - \tan A}{\cot A + \tan A}.$
18. $\sec 2a = \frac{\cot^2 a + 1}{\cot^2 a - 1}.$
19. $\operatorname{cosec} 2\theta = \frac{\cot^2 \theta + 1}{2 \cot \theta}.$
20. $\frac{1}{\cos A - \sin A} + \frac{1}{\cos A + \sin A} = 2 \cos A \sec 2A.$
21. $\frac{1}{1 - \tan A} - \frac{1}{1 + \tan A} = \tan 2A.$
22. $\sec 2A = 1 + \frac{\tan A}{1 - \tan A} - \frac{\tan A}{1 + \tan A}.$
23. $\sin 4\theta = 4 \sin \theta \cos \theta \cos 2\theta.$
24. $\cos 4A = 1 - 8 \sin^2 A \cos^2 A.$
25. $\frac{\cos A - \sin A}{\cos A + \sin A} = \frac{\cos 2A}{1 + \sin 2A}.$
26. $\tan \left(\frac{\pi}{4} - A \right) = \frac{\cos 2A}{1 + \sin 2A}.$
27. $\tan \left(A + \frac{3\pi}{4} \right) = \frac{\sin 2A - 1}{\cos 2A}.$
28. $\frac{1 - \cos \theta}{1 + \cos \theta} = \tan^2 \frac{\theta}{2}.$
29. $\cos 4\theta = (\cos 2\theta + \sin 2\theta)(\cos^2 \theta - 2 \sin \theta \cos \theta - \sin^2 \theta).$
30. $\cot 2\phi = \frac{\cot^2 \phi - 1}{2 \cot \phi}.$

31. $\frac{1 + \sin 2A}{1 + \cos 2A} = \frac{1}{2}(1 + \tan A)^2.$
32. $\sin 2a + 2 \sin a = 4 \sin a \cos^2 \frac{a}{2}.$
33. $\sin 2A + 2 \cos A = 2 \cos A \left(\cos \frac{A}{2} + \sin \frac{A}{2} \right)^2.$
34. $\frac{1 - \cos 2A}{1 - \sin 2A} = \frac{2}{(\cot A - 1)^2}.$
35. $\operatorname{cosec} 2\theta = \cot \theta - \cot 2\theta.$
36. $\tan 2\theta = \frac{\sin 4\theta + \sin 2\theta}{\cos 4\theta + 2 \cos^2 \theta}.$
37. $(1 + \tan A)(1 + \cot A) = 2(1 + \operatorname{cosec} 2A).$
38. $(\sec A - 1)(\operatorname{cosec} A - 1) = \tan \frac{A}{2} \cdot \frac{1 - \tan \frac{A}{2}}{1 + \tan \frac{A}{2}}.$
39. $\cos^4 \theta + \cos^3 \theta \sin \theta - \cos \theta \sin^3 \theta - \sin^4 \theta = \cos 2\theta + \frac{1}{4} \sin 4\theta.$
40. $\cos^4 \theta - \cos^3 \theta \sin \theta + \cos \theta \sin^3 \theta - \sin^4 \theta = \cos 2\theta - \frac{1}{4} \sin 4\theta.$
41. $(1 + \sin a)^2 = \frac{3}{2} + 2 \sin a - \frac{1}{2} \cos 2a.$
42. $(1 - \cos a)^2 = \frac{3}{2} - 2 \cos a + \frac{1}{2} \cos 2a.$
43. $\cos^4 a = \frac{3}{8} + \frac{1}{2} \cos 2a + \frac{1}{8} \cos 4a.$
44. $\sin^4 a = \frac{3}{8} - \frac{1}{2} \cos 2a + \frac{1}{8} \cos 4a.$
45. $\frac{\sin 2A + \cos 2A}{\sin 2A - \cos 2A} = \frac{1 + 2 \tan A - \tan^2 A}{\tan^2 A + 2 \tan A - 1}.$
46. $\sec 2\theta = \frac{\sec^2 \theta}{2 - \sec^2 \theta} = \frac{\operatorname{cosec}^2 \theta}{\operatorname{cosec}^2 \theta - 2}.$
47. $\cos 4a = 8 \cos^4 a - 8 \cos^2 a + 1 = 1 - 8 \sin^2 a + 8 \sin^4 a.$
48. $\tan 4x = \frac{4 \tan x - 4 \tan^3 x}{1 - 6 \tan^2 x + \tan^4 x}.$
49. $\cos^6 A - \sin^6 A = \cos 2A (\cos^2 2A + \frac{3}{4} \sin^2 2A).$
50. $\cos^6 A + \sin^6 A = \cos^2 2A + \frac{1}{4} \sin^2 2A.$
51. $\frac{\sin 2\theta + \cos \theta}{\cos 2\theta + \sin \theta} = \frac{\cos \theta}{1 - \sin \theta}.$
52. $\sin 3a + \sin a = 2 \sin 2a \cos a.$
53. $\cos a - \cos 3a = 2 \sin 2a \sin a.$
54. $\sin 6a - 3 \sin 2a = 4 \sin^3 (\pi + 2a).$
55. $\cos 3A - \sin 3A = (\cos A + \sin A)(1 - 2 \sin 2A).$
56. $2 (\sin 3\theta + \sin^3 \theta) = 3 \sin 2\theta \cos \theta.$
57. $2 (\cos^3 \theta - \cos 3\theta) = 3 \sin 2\theta \sin \theta.$
58. $\frac{1}{1 + \cos A} - \frac{1}{1 + \sin A} + \frac{1}{1 - \cos A} - \frac{1}{1 - \sin A} = 32 \cos^3 2A \operatorname{cosec}^2 4A.$

59. $\sin(A+B) \sin(A-B) = \frac{1}{2} (\cos 2B - \cos 2A)$.
60. $\cos(A+B) \cos(A-B) = \frac{1}{2} (\cos 2A + \cos 2B)$.
61. $2 \sin(A+B) \cos(A-B) = \sin 2A + \sin 2B$.
62. $2 \cos(A+B) \sin(A-B) = \sin 2A - \sin 2B$.
63. $\tan 6A = \frac{6 \tan A - 20 \tan^3 A + 6 \tan^5 A}{1 - 15 \tan^2 A + 15 \tan^4 A - \tan^6 A}$.
64. $\sin 2^n \theta = 2^n \sin \theta \cos \theta \cos 2\theta \cos 4\theta \dots \cos 2^{n-1} \theta$ where n is any positive integer.
65. $\cot \phi - 8 \cot 8\phi = \tan \phi + 2 \tan 2\phi + 4 \tan 4\phi$.
66. $\operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \operatorname{cosec} 8\theta = \cot \theta - \cot 8\theta$.
-
67. If θ is acute and $\cos \theta = 0.62$, show that $\cos \frac{\theta}{2} = 0.9$.
68. If $\sin \alpha = \frac{1}{3}$ and $\cos \alpha$ is negative, show that $\tan 2\alpha = -\frac{4\sqrt{2}}{7}$.
69. Prove $\tan 22\frac{1}{2}^\circ = \sqrt{2} - 1$.
70. Prove $\cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$.
71. Prove $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$.
72. Prove $\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$.
73. If $x = a \cos 2\theta$, $y = b \cos \theta$, prove that $b^2x = a(2y^2 - b^2)$.
74. If $x = a \tan \frac{\theta}{2}$, $y = b \sin \theta$, prove that $2abx = y(a^2 + x^2)$.
75. If $x = a \tan \phi$, $y = b \cos 2\phi$, prove that $b(a^2 - x^2) = y(a^2 + x^2)$.
76. If $x = p \tan \alpha$, $y = q \tan 2\alpha$, prove that $y(p^2 - x^2) = 2pqx$.
77. If $x = a \cos \theta$, $y = b \cos 3\theta$, prove that $a^3y = bx(4x^2 - 3a^2)$.
78. If $x = a \tan 3\theta$, $y = b \tan \theta$, prove that $bx(b^2 - 3y^2) = ay(3b^2 - y^2)$.

CHAPTER IX

PRODUCT FORMULÆ

- § 28. To prove that
- (i) $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
 - (ii) $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$
 - (iii) $2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
 - (iv) $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.

(i) and (ii) Since $\sin A \cos B + \cos A \sin B = \sin (A + B)$
 and $\sin A \cos B - \cos A \sin B = \sin (A - B)$
 $\therefore 2 \sin A \cos B = \sin (A + B) + \sin (A - B)$
 and $2 \cos A \sin B = \sin (A + B) - \sin (A - B)$.

(iii) and (iv) Since $\cos A \cos B - \sin A \sin B = \cos (A + B)$
 and $\cos A \cos B + \sin A \sin B = \cos (A - B)$
 $\therefore 2 \cos A \cos B = \cos (A + B) + \cos (A - B)$
 and $2 \sin A \sin B = \cos (A - B) - \cos (A + B)$.

Note 1. These relations are true for all values of A and B , since the expressions for $\sin (A \pm B)$ and $\cos (A \pm B)$ are true for all values of A and B .

Note 2. The relations enable us to express twice the product of the sine or cosine of one angle and the sine or cosine of another angle as the sum or difference of two circular functions.

Note 3. The product of a sine and a cosine is associated with the sum or difference of two sines; the product of two cosines or two sines is associated with the sum or difference of two cosines.

Note 4. While these relations are true for all values of A and B , it is convenient to look upon A as the larger angle, and to memorize the relations in words, thus :

- (i) Twice the sin of an angle $\times$ the cos of a smaller angle
 $= \sin (\text{sum of angles}) + \sin (\text{difference of angles}).$
- (ii) Twice the cos of an angle $\times$ the sin of a smaller angle
 $= \sin (\text{sum of angles}) - \sin (\text{difference of angles}).$
- (iii) Twice the product of two cosines
 $= \cos (\text{sum of angles}) + \cos (\text{difference of angles}).$

(iv) Twice the product of two sines

$$= \cos (\text{difference of angles}) - \cos (\text{sum of angles}).$$

Relation (iv) should be specially noted.

EXERCISES

Express as a sum or difference of two circular functions

1. $2 \sin 17^\circ \cos 8^\circ$.
2. $2 \cos 17^\circ \sin 8^\circ$.
3. $2 \sin 23^\circ \sin 6^\circ$.
4. $2 \cos 43^\circ \cos 23^\circ$.
5. $\cos 61^\circ \sin 25^\circ$.
6. $\sin 49^\circ \sin 22^\circ$.
7. $2 \sin 2A \cos A$.
8. $2 \cos 2A \cos A$.
9. $2 \sin \frac{A}{2} \cos \frac{3A}{2}$.
10. $2 \sin \frac{5A}{2} \sin \frac{3A}{2}$.
11. $2 \sin 2A \cos B$.
12. $\sin \theta \sin \phi$.
13. $2 \cos (3A + 2B) \sin (3B + 2A)$.
14. $2 \sin (x + y) \sin (x - y)$.
15. $2 \cos \frac{\theta - \phi}{2} \sin \frac{\theta + \phi}{2}$.
16. $2 \sin (2A + B) \cos (2A - B)$.
17. $2 \cos \frac{A + B}{2} \cos \frac{A + B + 2C}{2}$.
18. $2 \sin \frac{x - y}{2} \sin \frac{x - y + 2z}{2}$.
19. $\sin \left(\frac{\pi}{3} + \theta \right) \cos \left(\frac{\pi}{4} - \theta \right)$.
20. $\cos \left(\frac{2\pi}{3} + 2\phi \right) \cos \left(\frac{5\pi}{6} - 3\phi \right)$.

Prove that

21. $2 \sin 23^\circ \sin 22^\circ = \cos 1^\circ - \frac{1}{\sqrt{2}}$.
 22. $2 \sin 25^\circ \sin 35^\circ = \cos 10^\circ - \frac{1}{2}$.
 23. $2 \sin 45^\circ \cos 35^\circ = \cos 10^\circ + \sin 10^\circ$.
 24. $4 \cos 59^\circ \cos 61^\circ = 2 \cos 2^\circ - 1$.
 25. $2 \sin 3\theta \sin 4\theta + 2 \cos 5\theta \cos 2\theta - \cos 3\theta = \cos \theta$.
 26. $2 \sin 7\theta \cos 5\theta + 2 \cos 5\theta \sin 3\theta - \sin 8\theta = 4 \sin 3\theta \cos 3\theta \cos 6\theta$.
 27. $\sin 5A \sin 7A + \sin 2A \sin 6A - \sin 3A \sin 5A$
 $= 2 \cos 4A - 2 \cos^3 4A$.
 28. $2 \sin \frac{\theta}{2} \cos \frac{3\theta}{2} + 2 \sin \frac{5\theta}{2} \cos \frac{3\theta}{2} + 2 \cos \frac{7\theta}{2} \sin \frac{3\theta}{2} = \sin 5\theta + \sin 4\theta$.
- § 29. To prove that
- (i) $\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}$
 - (ii) $\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$
 - (iii) $\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}$
 - (iv) $\cos D - \cos C = 2 \sin \frac{C + D}{2} \sin \frac{C - D}{2}$.

The relations of § 28 may be written

$$\sin (A+B)+\sin (A-B)=2 \sin A \cos B$$

$$\sin (A+B)-\sin (A-B)=2 \cos A \sin B$$

$$\cos (A+B)+\cos (A-B)=2 \cos A \cos B$$

$$\cos (A-B)-\cos (A+B)=2 \sin A \sin B$$

Letting $A+B=C$ and $A-B=D$

$$\text{we have } A=\frac{C+D}{2} \text{ and } B=\frac{C-D}{2}$$

Hence

$$\sin C+\sin D=2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\sin C-\sin D=2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}$$

$$\cos C+\cos D=2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}$$

$$\cos D-\cos C=2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}.$$

Note 1. These relations are true for all values of C and D , since the relations of § 28 are true for all values of A and B .

Note 2. These relations enable us to express the sum or difference of two sines or two cosines as twice the product of two circular functions. (Compare § 28, *Note 2*.)

Note 3. The sum or difference of two sines is associated with the product of a sine and a cosine: the sum or difference of two cosines is associated with the product of two cosines or two sines.

Note 4. While these relations are true for all values of C and D , it is convenient to look upon C as the larger angle, and to memorize the relations in words, thus:

(i) $\sin + \sin = \text{twice } \sin (\text{half sum}) \cos (\text{half difference}).$

(ii) $\sin (\text{large angle}) - \sin (\text{small angle})$
 $= \text{twice } \cos (\text{half sum}) \sin (\text{half difference}).$

(iii) $\cos + \cos = \text{twice } \cos (\text{half sum}) \cos (\text{half difference}).$

(iv) $\cos (\text{small angle}) - \cos (\text{large angle})$
 $= \text{twice } \sin (\text{half sum}) \sin (\text{half difference}).$

Relation (iv) should be specially noted.

§ 30. GEOMETRICAL PROOF OF PRODUCT FORMULÆ

Let the acute angle $\angle XOC = \hat{C}$ (Fig. 57), and let the smaller acute angle $\angle XOD = \hat{D}$.

Draw OE bisecting $\hat{DOC}$ and draw HPK perpendicular to OE .

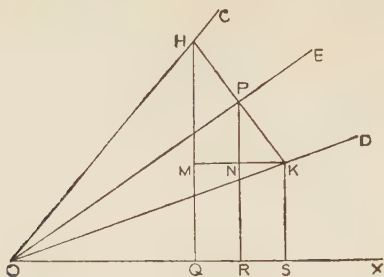


Fig. 57

Draw HQ , PR , KS perpendicular to OX , and KNM perpendicular to HQ .

$$\text{Then } \hat{DOE} = \frac{C - D}{2}$$

$$\hat{XOE} = \hat{XOD} + \hat{DOE} = D + \frac{C - D}{2} = \frac{C + D}{2}$$

$HP = PK$ and $OH = OK$ (Δ 's OPH , OPK congruent)

$MN = NK$ and $QR = RS$ ($HP = PK$)

$$2RP = 2RN + 2NP = SK + QM + MH = SK + QH$$

and $\hat{RPK} = \hat{ROP}$ (each is complement of $\hat{OPR}$)

$$\begin{aligned} \therefore (i) \sin C + \sin D &= \frac{QH}{OH} + \frac{SK}{OK} \\ &= \frac{QH + SK}{OK} \\ &= 2 \frac{RP}{OK} \\ &= 2 \frac{RP}{OP} \cdot \frac{OP}{OK} \\ &= 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}. \end{aligned}$$

$$\begin{aligned} (ii) \sin C - \sin D &= \frac{QH}{OH} - \frac{SK}{OK} \\ &= \frac{QH - SK}{OK} \\ &= 2 \frac{NP}{OK} \end{aligned}$$

$$= 2 \frac{NP}{PK} \cdot \frac{PK}{OK}$$

$$= 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2}.$$

$$(iii) \cos C + \cos D = \frac{OQ}{OH} + \frac{OS}{OK}$$

$$= \frac{OQ + OS}{OK}$$

$$= 2 \frac{OR}{OK}$$

$$= 2 \frac{OR}{OP} \cdot \frac{OP}{OK}$$

$$= 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2}.$$

$$(iv) \cos D - \cos C = \frac{OS}{OK} - \frac{OQ}{OH}$$

$$= \frac{OS - OQ}{OK}$$

$$= 2 \frac{RS}{OK}$$

$$= 2 \frac{NK}{PK} \cdot \frac{PK}{OK}$$

$$= 2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}.$$

Example 1. Prove that

$$\frac{\cos 2\theta + \sin 4\theta - \cos 6\theta}{\sin 6\theta + \cos 4\theta - \sin 2\theta} = \tan 4\theta.$$

$$\frac{\cos 2\theta + \sin 4\theta - \cos 6\theta}{\sin 6\theta + \cos 4\theta - \sin 2\theta} = \frac{2 \sin 4\theta \sin 2\theta + \sin 4\theta}{2 \cos 4\theta \sin 2\theta + \cos 4\theta}$$

$$= \frac{\sin 4\theta (2 \sin 2\theta + 1)}{\cos 4\theta (2 \sin 2\theta + 1)}$$

$$= \tan 4\theta.$$

Example 2. Prove that

$$\sin (A + 45^\circ) - \sin (A - 45^\circ) + \cos (A + 135^\circ) + \cos (A - 135^\circ) = 0.$$

$$\sin (A + 45^\circ) - \sin (A - 45^\circ) + \cos (A + 135^\circ) + \cos (A - 135^\circ)$$

$$= 2 \cos A \sin 45^\circ + 2 \cos A \cos 135^\circ$$

$$= 2 \cos A \sin 45^\circ - 2 \cos A \cos 45^\circ$$

$$= 0.$$

Example 3. Prove that

$$\sin (3\theta + \phi) - \sin (\theta - 3\phi) - 2 \cos (4\theta - \phi) \sin (\theta + 2\phi)$$

$$= 4 \sin (\theta + 2\phi) \sin (3\theta - \phi) \sin \theta.$$

$$\begin{aligned}
& \sin(3\theta + \phi) - \sin(\theta - 3\phi) - 2 \cos(4\theta - \phi) \sin(\theta + 2\phi) \\
&= 2 \cos(2\theta - \phi) \sin(\theta + 2\phi) - 2 \cos(4\theta - \phi) \sin(\theta + 2\phi) \\
&= 2 \sin(\theta + 2\phi) [\cos(2\theta - \phi) - \cos(4\theta - \phi)] \\
&= 4 \sin(\theta + 2\phi) \sin(3\theta - \phi) \sin \theta.
\end{aligned}$$

EXERCISES

Express as twice the product of two circular functions

- | | |
|-----------------------------------|---|
| 1. $\sin 3A + \sin A.$ | 9. $\cos \frac{5A}{2} - \cos \frac{7A}{2}.$ |
| 2. $\cos 4A - \cos 8A.$ | 10. $\sin A + \cos A.$ |
| 3. $\cos 7A + \cos 3A.$ | 11. $\sin 37^\circ + \sin 25^\circ.$ |
| 4. $\sin 5A - \sin 2A.$ | 12. $\cos 5^\circ - \cos 59^\circ.$ |
| 5. $\sin 5\theta + \sin 7\theta.$ | 13. $\cos 33^\circ + \cos 23^\circ.$ |
| 6. $\cos 6\theta + \cos 9\theta.$ | 14. $\sin 64^\circ - \sin 54^\circ.$ |
| 7. $\cos 2A - \cos 6A.$ | 15. $\cos 84^\circ - \sin 4^\circ.$ |
| 8. $\sin(A + 3B) - \sin(3A + B).$ | 16. $\cos 52^\circ + \sin 19^\circ.$ |

Prove that

- | | |
|---|--|
| 17. $\cos 21^\circ + \cos 39^\circ = \sqrt{3} \cos 9^\circ.$ | 24. $\frac{\sin 3\theta - \sin \theta}{\cos 3\theta + \cos \theta} = \tan \theta.$ |
| 18. $\sin 57^\circ - \sin 33^\circ = \sqrt{2} \sin 12^\circ.$ | 25. $\frac{\cos \theta - \cos 3\theta}{\sin \theta + \sin 3\theta} = \tan \theta.$ |
| 19. $\frac{\cos 75^\circ + \cos 15^\circ}{\sin 105^\circ - \sin 15^\circ} = \sqrt{3}.$ | 26. $\frac{\sin 5\theta - \sin 3\theta}{\cos 5\theta + \cos 3\theta} = \tan \theta.$ |
| 20. $\frac{\sin 53^\circ - \sin 7^\circ}{\cos 7^\circ - \cos 53^\circ} = \sqrt{3}.$ | 27. $\frac{\sin 5\theta + \sin 3\theta}{\cos 3\theta - \cos 5\theta} = \cot \theta.$ |
| 21. $\frac{\cos 25^\circ - \cos 35^\circ}{\sin 35^\circ - \sin 25^\circ} = \frac{1}{\sqrt{3}}.$ | 28. $\frac{\sin 2A - \sin A}{\cos A - \cos 2A} = \cot \frac{3A}{2}.$ |
| 22. $\frac{\sin 243^\circ - \sin 27^\circ}{\cos 27^\circ - \cos 243^\circ} = -1.$ | 29. $\frac{\sin 3\phi + \sin 7\phi}{\cos 3\phi + \cos 7\phi} = \tan 5\phi.$ |
| 23. $\frac{\cos 3\theta + \cos \theta}{\sin 3\theta + \sin \theta} = \cot 2\theta.$ | 30. $\frac{\cos 4A + \cos 5A}{\sin 5A - \sin 4A} = \cot \frac{A}{2}.$ |
| 31. $\frac{\sin A - \sin B}{\cos B - \cos A} = \cot \frac{A+B}{2}.$ | |
| 32. $\frac{\sin(x+y) - \sin(x-y)}{\sin(x+y) + \sin(x-y)} = \cot x \tan y.$ | |
| 33. $\frac{\sin(x+y) + \sin(x-y)}{\cos(x-y) - \cos(x+y)} = \cot y.$ | |
| 34. $\frac{\cos(2\theta - \phi) - \cos(2\theta + \phi)}{\sin(2\theta + \phi) - \sin(2\theta - \phi)} = \tan 2\theta.$ | |

$$35. \frac{\sin (3A - B) - \sin (A - 3B)}{\cos (3A - B) + \cos (A - 3B)} = \tan (A + B).$$

$$36. \frac{\cos \frac{\pi}{6} - \cos \frac{\pi}{3}}{\sin \frac{\pi}{6} + \sin \frac{\pi}{3}} = \tan \frac{\pi}{12} = 2 - \sqrt{3}.$$

$$37. \frac{\sin \frac{2\pi}{3} - \sin \frac{\pi}{4}}{\cos \frac{2\pi}{3} + \cos \frac{\pi}{4}} = \tan \frac{5\pi}{24} = \sqrt{6} + \sqrt{3} - \sqrt{2} - 2.$$

$$38. \frac{\cos \frac{\pi}{4} + \cos \frac{5\pi}{6}}{\sin \frac{\pi}{4} + \sin \frac{5\pi}{6}} = -\tan \frac{\pi}{24} = 2 - \sqrt{2} + \sqrt{3} - \sqrt{6}.$$

$$39. \cos A + \cos (A + 120^\circ) + \cos (A + 240^\circ) = 0.$$

$$40. \sin A - \sin (A + 60^\circ) + \sin (A + 120^\circ) = 0.$$

$$41. \cos A - \sin (A + 150^\circ) + \sin (A - 150^\circ) = 0.$$

$$42. \cos A - \sin (A + 30^\circ) + \sin (A + 330^\circ) = 0.$$

$$43. \cos (A + 225^\circ) + \cos (A - 225^\circ) + \sin (A - 315^\circ) - \sin (A + 315^\circ) = 0.$$

$$44. \sin (A + 210^\circ) + \sin (A - 210^\circ) - \cos (A - 300^\circ) + \cos (A + 300^\circ) = 0.$$

$$45. \sin 3\theta \sin 4\theta + \cos 5\theta \cos 2\theta = \cos 2\theta \cos \theta.$$

$$46. \sin 7\theta \cos 5\theta + \cos 5\theta \sin 3\theta = \sin 10\theta \cos 2\theta.$$

$$47. \sin 5\theta + \sin 3\theta - 2 \sin 2\theta \cos \theta = 4 \sin \theta \cos \theta \cos 3\theta.$$

$$48. \cos 3A - \cos 7A + 2 \sin 2A \sin 3A = 4 \sin 2A \sin 4A \cos A.$$

$$49. \frac{\cos 2A - \cos 4A + \cos 6A}{\sin 2A - \sin 4A + \sin 6A} = \cot 4A.$$

$$50. \frac{\cos 3A - \sin 6A - \cos 9A}{\sin 9A - \cos 6A - \sin 3A} = \tan 6A.$$

$$51. \frac{\sin 5A - \sin 6A - \sin 8A + \sin 9A}{\cos 5A - \cos 6A - \cos 8A + \cos 9A} = \tan 7A.$$

$$52. \frac{\sin 2\theta + \sin \theta - \cos \theta + \cos 2\theta}{\cos \theta + \cos 2\theta - \sin 2\theta + \sin \theta} = \tan \frac{3\theta}{2}.$$

$$53. \frac{\sin 2A \sin A + \sin 5A \sin 2A + \sin 11A \sin 4A}{\cos 2A \sin A - \cos 2A \sin 5A + \cos 4A \sin 11A} = \tan 8A.$$

$$54. \frac{\cos \frac{3\theta}{2} \cos \frac{\theta}{2} - \sin \frac{5\theta}{2} \sin \frac{3\theta}{2} - \sin \frac{13\theta}{2} \sin \frac{5\theta}{2}}{\sin \frac{3\theta}{2} \cos \frac{\theta}{2} - \cos \frac{3\theta}{2} \sin \frac{5\theta}{2} + \cos \frac{5\theta}{2} \sin \frac{13\theta}{2}} = \cot \frac{11\theta}{2}.$$

$$55. 2 \sin \frac{2\pi}{9} \cos \frac{4\pi}{27} + 2 \sin \frac{5\pi}{27} \cos \frac{5\pi}{9} = \sin \frac{11\pi}{27}.$$

$$56. \quad 2 \cos \frac{4\pi}{15} \sin \frac{\pi}{30} + 2 \sin \frac{2\pi}{15} \cos \frac{13\pi}{30} = \cos \frac{2\pi}{5}.$$

$$57. \quad \text{If } \cos \beta = k \cos \alpha, \text{ prove that } \tan \frac{\alpha - \beta}{2} = \frac{k - 1}{k + 1} \cot \frac{\alpha + \beta}{2}.$$

$$58. \quad \text{If } \sin 2\theta = k \sin 2\phi, \text{ prove that } (k - 1) \tan (\theta + \phi) = (k + 1) \tan (\theta - \phi).$$

$$59. \quad \text{If } x = \frac{\cos A - \sin B}{\sin A - \cos B}, \text{ prove that } (x - 1) \tan \frac{A - B}{2} = x + 1.$$

$$60. \quad \text{If } k = \frac{\sin 2\theta - \cos 2\phi}{\cos 2\theta - \sin 2\phi}, \text{ prove that } \frac{1 - k}{1 + k} = \cot (\theta - \phi).$$

§ 31. The following examples give good practice in the use of the product formulæ.

Example 1. In any triangle ABC prove that

$$\cos A - \cos B + \cos C = 4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} - 1.$$

$$\begin{aligned} \text{1st proof} \quad \cos A - \cos B + \cos C &= 2 \sin \frac{B+A}{2} \sin \frac{B-A}{2} + 2 \cos^2 \frac{C}{2} - 1 \\ &= 2 \cos \frac{C}{2} \left(\sin \frac{B-A}{2} + \sin \frac{B+A}{2} \right) - 1 \\ &= 4 \cos \frac{C}{2} \cdot \sin \frac{B}{2} \cos \frac{A}{2} - 1 \\ &= 4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} - 1. \end{aligned}$$

$$\begin{aligned} \text{2nd proof} \quad 4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} - 1 &= 2 \cos \frac{A}{2} \cdot 2 \sin \frac{B}{2} \cos \frac{C}{2} - 1 \\ &= 2 \cos \frac{A}{2} \left(\sin \frac{B+C}{2} + \sin \frac{B-C}{2} \right) - 1 \\ &= 2 \cos^2 \frac{A}{2} - 1 + 2 \sin \frac{B+C}{2} \sin \frac{B-C}{2} \\ &= \cos A + \cos C - \cos B \\ &= \cos A - \cos B + \cos C. \end{aligned}$$

Example 2. If $A + B + C = 180^\circ$, prove that

$$\sin^2 B + \sin^2 C - \sin^2 A = 2 \cos A \sin B \sin C.$$

$$\begin{aligned} \sin^2 B + \sin^2 C - \sin^2 A &= \frac{1}{2} [2 \sin^2 B + 2 \sin^2 C - 2 \sin^2 A] \\ &= \frac{1}{2} [1 - \cos 2B + 2 \sin^2 C + \cos 2A - 1] \\ &= \frac{1}{2} [\cos 2A - \cos 2B + 2 \sin^2 C] \\ &= \frac{1}{2} [2 \sin (B + A) \sin (B - A) + 2 \sin^2 C] \end{aligned}$$

$$\begin{aligned} &= \sin C [\sin (B - A) + \sin (B + A)] \\ &= 2 \sin C \cdot \sin B \cos A \\ &= 2 \cos A \sin B \sin C. \end{aligned}$$

Example 3. Prove that

$$\begin{aligned} \sin x - \sin y - \sin z + \sin (y + z - x) &= 4 \sin \frac{x-y}{2} \sin \frac{y+z}{2} \sin \frac{z-x}{2}. \\ \sin x - \sin y - \sin z + \sin (y + z - x) \\ &= 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2} - 2 \cos \frac{y+2z-x}{2} \sin \frac{x-y}{2} \\ &= 2 \sin \frac{x-y}{2} \left(\cos \frac{x+y}{2} - \cos \frac{y+2z-x}{2} \right) \\ &= 4 \sin \frac{x-y}{2} \sin \frac{y+z}{2} \sin \frac{z-x}{2}. \end{aligned}$$

EXERCISES

Prove that in any triangle ABC

1. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$
2. $\sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$
3. $\cos A + \cos B + \cos C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1.$
4. $\cos A + \cos B - \cos C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} - 1.$
5. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$
6. $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C.$
7. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$
8. $\cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C.$
9. $\sin 4A + \sin 4B + \sin 4C = -4 \sin 2A \sin 2B \sin 2C.$
10. $\sin 4A + \sin 4B - \sin 4C = -4 \cos 2A \cos 2B \sin 2C.$
11. $\cos 4A + \cos 4B + \cos 4C = 4 \cos 2A \cos 2B \cos 2C - 1.$
12. $\cos 4A + \cos 4B - \cos 4C = 4 \sin 2A \sin 2B \cos 2C + 1.$
13. $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$
14. $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2} = 1 - 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$
15. $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$
16. $\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} - \cos^2 \frac{C}{2} = 2 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$
17. $\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2 \cos A \cos B \cos C.$

18. $\sin^2 A + \sin^2 B - \sin^2 C = 2 \sin A \sin B \cos C.$
19. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$
20. $\cos^2 A + \cos^2 B - \cos^2 C = 1 - 2 \sin A \sin B \cos C.$
21. $\sin^2 2A + \sin^2 2B + \sin^2 2C = 2 - 2 \cos 2A \cos 2B \cos 2C.$
22. $\sin^2 2A + \sin^2 2B - \sin^2 2C = -2 \sin 2A \sin 2B \cos 2C.$
23. $\cos^2 2A + \cos^2 2B + \cos^2 2C = 1 + 2 \cos 2A \cos 2B \cos 2C.$
24. $\cos^2 2A + \cos^2 2B - \cos^2 2C = 1 + 2 \sin 2A \sin 2B \cos 2C.$
25. $\tan A + \tan B + \tan C = \tan A \tan B \tan C.$
26. $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1.$
27. $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1.$
28. $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}.$
29. $\tan \frac{A}{4} + \tan \frac{B}{4} + \tan \frac{C}{4} + \tan \frac{A}{4} \tan \frac{B}{4} + \tan \frac{B}{4} \tan \frac{C}{4} + \tan \frac{C}{4} \tan \frac{A}{4}$
 $= \tan \frac{A}{4} \tan \frac{B}{4} \tan \frac{C}{4}.$
30. $\tan \frac{A}{2} \tan \frac{B}{2} \cot \frac{C}{2} + \tan \frac{B}{2} \tan \frac{C}{2} \cot \frac{A}{2} + \tan \frac{C}{2} \tan \frac{A}{2} \cot \frac{B}{2}$
 $= \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} - 2 \left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)$
31. $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} = 1 + 4 \cos \frac{\pi + A}{4} \cos \frac{\pi + B}{4} \cos \frac{\pi + C}{4}.$
32. $\cos \frac{A}{2} + \cos \frac{B}{2} + \cos \frac{C}{2} = 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$
33. $2 \left(\sin^2 \frac{A}{4} + \sin^2 \frac{B}{4} + \sin^2 \frac{C}{4} \right) = 3 - 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$
34. $2 \left(\cos^2 \frac{A}{4} + \cos^2 \frac{B}{4} + \cos^2 \frac{C}{4} \right) = 3 + 4 \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}.$
35. $\frac{\sin A + \sin B - \sin C}{\sin A - \sin B + \sin C} = \tan \frac{B}{2} \cot \frac{C}{2}.$
36. $\frac{\cos A + \cos B + \cos C - 1}{\cos B + \cos C - \cos A + 1} = \tan \frac{B}{2} \tan \frac{C}{2}.$
37. $\frac{\sin 2A - \sin 2B + \sin 2C}{\sin^2 A - \sin^2 B + \sin^2 C} = 2 \cot A \tan B \cot C.$
38. $\frac{1 - \sin^2 \frac{A}{2} - \sin^2 \frac{B}{2} - \sin^2 \frac{C}{2}}{\cos^2 \frac{A}{2} - \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}} = \tan \frac{C}{2} \tan \frac{A}{2}.$
39. $\frac{\sin 4A - \sin 4B + \sin 4C}{\sin 4B + \sin 4C - \sin 4A} = \frac{1 + \cos 4A - \cos 4B - \cos 4C}{1 - \cos 4A + \cos 4B - \cos 4C}.$

Prove that

40. $\sin x + \sin y + \sin z + \sin (x - y + z) = 4 \cos \frac{x-y}{2} \cos \frac{y-z}{2} \sin \frac{z+x}{2}.$
41. $\sin x + \sin y + \sin z + \sin (y + z - x) = 4 \cos \frac{z-x}{2} \cos \frac{x-y}{2} \sin \frac{y+z}{2}.$
42. $\cos x + \cos y + \cos z + \cos (x - y + z) = 4 \cos \frac{x-y}{2} \cos \frac{y-z}{2} \cos \frac{z+x}{2}.$
43. $\cos x + \cos y + \cos z + \cos (x + y - z) = 4 \cos \frac{y-z}{2} \cos \frac{z-x}{2} \cos \frac{x+y}{2}.$
44. $\sin x - \sin y + \sin z - \sin (x - y + z) = -4 \sin \frac{x-y}{2} \sin \frac{y-z}{2} \sin \frac{z+x}{2}.$
45. $\cos x - \cos y + \cos z - \cos (x - y + z) = -4 \sin \frac{x-y}{2} \sin \frac{y-z}{2} \cos \frac{z+x}{2}.$
46. $\sin^2 \frac{x}{2} - \sin^2 \frac{y}{2} + \sin^2 \frac{z}{2} - \sin^2 \left(\frac{x-y+z}{2} \right)$
 $= 2 \sin \frac{x-y}{2} \sin \frac{y-z}{2} \cos \frac{z+x}{2}.$
47. $\cos^2 \frac{x}{2} - \cos^2 \frac{y}{2} + \cos^2 \frac{z}{2} - \cos^2 \left(\frac{x-y+z}{2} \right)$
 $= -2 \sin \frac{x-y}{2} \sin \frac{y-z}{2} \cos \frac{z+x}{2}.$
48. $\sin 2x + \sin 2y + \sin 2z - \sin 2(x + y + z)$
 $= 4 \sin (x + y) \sin (y + z) \sin (z + x).$
49. $\cos 2x + \cos 2y + \cos 2z + \cos 2(x + y + z)$
 $= 4 \cos (x + y) \cos (y + z) \cos (z + x).$
50. $\sin^2 x + \sin^2 y + \sin^2 z + \sin^2 (x + y + z)$
 $= 2 - 2 \cos (x + y) \cos (y + z) \cos (z + x).$
51. $\cos^2 x + \cos^2 y + \cos^2 z + \cos^2 (x + y + z)$
 $= 2 + 2 \cos (x + y) \cos (y + z) \cos (z + x).$
52. $\sin (x - y) + \sin (y - z) + \sin (z - x) = -4 \sin \frac{x-y}{2} \sin \frac{y-z}{2} \sin \frac{z-x}{2}.$
53. $\cos (x - y) + \cos (y - z) + \cos (z - x)$
 $= 4 \cos \frac{x-y}{2} \cos \frac{y-z}{2} \cos \frac{z-x}{2} - 1.$
54. $\tan (x - y) + \tan (y - z) + \tan (z - x)$
 $= \tan (x - y) \tan (y - z) \tan (z - x).$
55. $\sin (x + y - z) + \sin (y + z - x) + \sin (z + x - y) - \sin (x + y + z)$
 $= 4 \sin x \sin y \sin z.$
56. $\cos (x + y - z) + \cos (y + z - x) + \cos (z + x - y) + \cos (x + y + z)$
 $= 4 \cos x \cos y \cos z.$

(The equations of § 54 may be taken at this stage.)

CHAPTER X

THE SIDES AND ANGLES OF ANY TRIANGLE

§ 32. NOTATION.

In this chapter we shall use the notation of Chapter VI, and, further, we shall denote

the number of units in the circumradius of a triangle by R
 the number of square units in the area of a triangle by Δ
 and the number of units in the perimeter of a triangle by $2s$.

Note 1. Since the perimeter is denoted by $2s$, s is called the semi-perimeter.

Note 2.

$$\text{Since } 2s = a + b + c$$

$$2s - 2a = b + c - a$$

$$\therefore s - a = \frac{b + c - a}{2}$$

$$\text{and similarly } s - b = \frac{c + a - b}{2}$$

$$\text{and } s - c = \frac{a + b - c}{2}.$$

§ 33. To prove that in any triangle $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

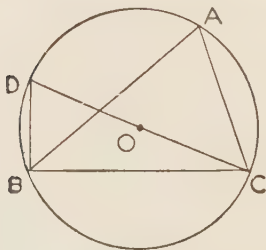


FIG. 58

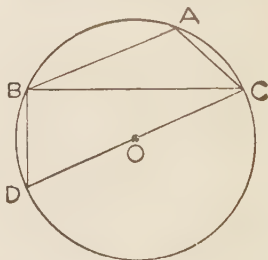


FIG. 59

Let O (Figs. 58, 59) be the circumcentre of the triangle ABC .

Join CO and produce it to meet the circumference of the circumcircle in D; join BD. In Fig. 58, $\hat{A}$ is acute; in Fig. 59, $\hat{A}$ is obtuse.

Then $\sin A = \sin D$ ($\hat{A} = \hat{D}$ in Fig. 58, $\hat{A} = 180^\circ - \hat{D}$ in Fig. 59)

$$= \frac{BC}{DC} \quad (\angle BDC = 90^\circ)$$

$$= \frac{a}{2R}$$

$$\therefore \frac{a}{\sin A} = 2R$$

Similarly $\frac{b}{\sin B} = 2R$, $\frac{c}{\sin C} = 2R$

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \quad . \quad . \quad . \quad . \quad . \quad (i)$$

Note 1. If $A = 90^\circ$, BC is a diameter of the circumcircle, and

$$\sin A = 1 = \frac{BC}{2R} = \frac{a}{2R}$$

$$\therefore \frac{a}{\sin A} = 2R.$$

Hence the relations (i) are true in all cases.

Note 2. The relations $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ can be written

$$a \sin B = b \sin A, \quad b \sin C = c \sin B, \quad c \sin A = a \sin C.$$

§ 34. To prove that in any triangle $a = b \cos C + c \cos B$
 $b = c \cos A + a \cos C$
 and $c = a \cos B + b \cos A$.

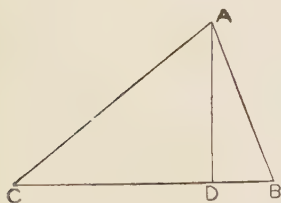


FIG. 60

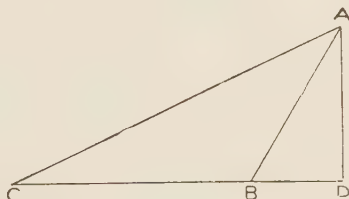


FIG. 61

Let AD (Figs. 60, 61), perpendicular to CB, meet CB or CB produced in D.

$$\begin{aligned} \text{In Fig. 60, } a &= CD + DB \\ &= b \cos C + c \cos B \end{aligned}$$

In Fig. 61, $a = CD - BD$

$$= b \cos C - c \cos \hat{A}BD$$

$$= b \cos C + c \cos B \quad (\hat{ABD} = 180^\circ - B).$$

$$\therefore \text{in both cases, } a = b \cos C + c \cos B \left. \begin{array}{l} \text{Similarly } b = c \cos A + a \cos C \\ \text{and } c = a \cos B + b \cos A \end{array} \right\} \quad \text{. . . (ii)}$$

Note 1. The student should verify that the relations (ii) still hold if D lies on BC produced, instead of on CB produced as in Fig. 61.

Note 2. If D coincides with B , the angle B is a right angle, and

$$a = b \cos C$$

$$= b \cos C + c \cos B \quad (\cos 90^\circ = 0).$$

Similarly, if D coincides with C , $\cos C = 0$ and $a = b \cos C + c \cos B$. Hence the relations (ii) are true in all cases.

§ 35. To prove that in any triangle $a^2 = b^2 + c^2 - 2bc \cos A$
 $b^2 = c^2 + a^2 - 2ca \cos B$
 and $c^2 = a^2 + b^2 - 2ab \cos C$.

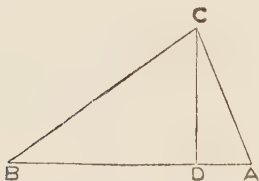


FIG. 62

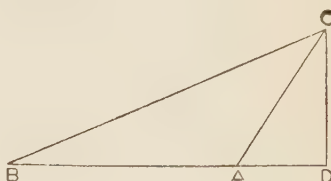


FIG. 63

Let CD (Figs. 62, 63), perpendicular to BA , meet BA or BA produced in D .

In Fig. 62, $\hat{A}$ is acute

$$\therefore BC^2 = CA^2 + AB^2 - 2AB \cdot AD$$

$$\text{i.e., } a^2 = b^2 + c^2 - 2c \cdot b \cos A.$$

In Fig. 63, $\hat{A}$ is obtuse

$$\therefore BC^2 = CA^2 + AB^2 + 2AB \cdot AD$$

$$\text{i.e., } a^2 = b^2 + c^2 + 2c \cdot b \cos \hat{D}AC$$

$$= b^2 + c^2 - 2c \cdot b \cos A \quad (\hat{DAC} = 180^\circ - A).$$

$$\therefore \text{in both cases, } a^2 = b^2 + c^2 - 2bc \cos A \left. \begin{array}{l} \text{Similarly } b^2 = c^2 + a^2 - 2ca \cos B \\ \text{and } c^2 = a^2 + b^2 - 2ab \cos C \end{array} \right\} \quad \text{. . . (iii)}$$

$$\begin{aligned}
 &= \frac{73}{80} \\
 &= 0.9125
 \end{aligned}$$

∴ the size of the smallest angle is $24^\circ 8'$.

EXERCISES

1. If $c=10$, $A=60^\circ$, and $B=45^\circ$ find a .
2. If $R=10$ and $A=30^\circ$ find a .
3. If $a=5$ and $A=60^\circ$ find R .
4. If $b=10$, $c=4$, and $A=60^\circ$ find a .
5. If $c=5$, $a=6$, and $B=60^\circ$ find b .
6. If $a=10$, $A=51^\circ$, and $C=83^\circ$ find b .
7. If $a=11.2$, $b=13.6$, and $C=57^\circ 30'$ find c .
8. If $a=4$, $b=5$, and $c=6$ find A .
9. If $a=5$, $b=6$, and $c=7$ find the largest angle.

Prove that in any triangle

10. $a \cos A = R \sin 2A$.
11. $a (\sin B + \sin C) = (b + c) \sin A$.
12. $a (\sin B - \sin C) + b (\sin C - \sin A) + c (\sin A - \sin B) = 0$.
13. $a \cos A \sin 2B = b \cos B \sin 2A$.
14. $\frac{\cos A}{a} + \frac{\cos B}{b} = \frac{c}{ab}$.
15. $\cos B (b - c \cos A) = \cos C (c - b \cos A)$.
16. $(a - b) (1 + \cos C) = c (\cos B - \cos A)$.
17. $a (\cos B + \cos C) = (b + c) (1 - \cos A)$.
18. $(b + c) \cos A + (c + a) \cos B + (a + b) \cos C = a + b + c$.
19. $a^2 + b^2 + c^2 = 2 (bc \cos A + ca \cos B + ab \cos C)$.
20. $3c^2 - a^2 - b^2 = 2 (bc \cos A + ca \cos B - ab \cos C)$.
21. $A = 60^\circ$, if $a^2 - c^2 = b (b - c)$.

§ 36. To prove that in any triangle $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

$$\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$$

and $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$.

$$\begin{aligned}
 \sin^2 \frac{A}{2} &= \frac{1}{2} (1 - \cos A) \\
 &= \frac{1}{2} \left(1 - \frac{b^2 + c^2 - a^2}{2bc} \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2bc - b^2 - c^2 + a^2}{4bc} \\
 &= \frac{a^2 - (b - c)^2}{4bc} \\
 &= \frac{(a - b + c)(a + b - c)}{4bc} \\
 &= \frac{2(s - b) \cdot 2(s - c)}{4bc} \\
 &= \frac{(s - b)(s - c)}{bc}
 \end{aligned}$$

$$\therefore \sin \frac{A}{2} = \pm \sqrt{\frac{(s - b)(s - c)}{bc}}.$$

But A , being an angle in a triangle, is less than 180° ; therefore $\frac{A}{2} < 90^\circ$ and $\sin \frac{A}{2}$ is positive.

$$\begin{aligned}
 \therefore \sin \frac{A}{2} &= \sqrt{\frac{(s - b)(s - c)}{bc}} \\
 \text{Similarly } \sin \frac{B}{2} &= \sqrt{\frac{(s - c)(s - a)}{ca}} \\
 \text{and } \sin \frac{C}{2} &= \sqrt{\frac{(s - a)(s - b)}{ab}}
 \end{aligned} \quad \text{. . . (v)}$$

Note. Since the relations (iv) are true for any triangle, the relations (v) are true for any triangle.

$$\begin{aligned}
 \S 37. \text{ To prove that in any triangle } \cos \frac{A}{2} &= \sqrt{\frac{s(s - a)}{bc}} \\
 \cos \frac{B}{2} &= \sqrt{\frac{s(s - b)}{ca}} \\
 \text{and } \cos \frac{C}{2} &= \sqrt{\frac{s(s - c)}{ab}}.
 \end{aligned}$$

$$\begin{aligned}
 \cos^2 \frac{A}{2} &= \frac{1}{2} (1 + \cos A) \\
 &= \frac{1}{2} \left(1 + \frac{b^2 + c^2 - a^2}{2bc} \right) \\
 &= \frac{2bc + b^2 + c^2 - a^2}{4bc} \\
 &= \frac{(b + c)^2 - a^2}{4bc} \\
 &= \frac{(b + c + a)(b + c - a)}{4bc}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2s \cdot 2(s-a)}{4bc} \\
 &= \frac{s(s-a)}{bc} \\
 \therefore \cos \frac{A}{2} &= \pm \sqrt{\frac{s(s-a)}{bc}}.
 \end{aligned}$$

But A , being an angle in a triangle, is less than 180° ; therefore $\frac{A}{2} < 90^\circ$ and $\cos \frac{A}{2}$ is positive.

$$\left. \begin{aligned}
 \therefore \cos \frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}} \\
 \text{Similarly } \cos \frac{B}{2} &= \sqrt{\frac{s(s-b)}{ca}} \\
 \text{and } \cos \frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}}
 \end{aligned} \right\} \quad . \quad . \quad . \quad (\text{vi})$$

Note. Since the relations (iv) are true for any triangle, the relations (vi) are true for any triangle.

§ 38. To prove that in any triangle $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$$\begin{aligned}
 \tan \frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \\
 \text{and } \tan \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.
 \end{aligned}$$

$$\begin{aligned}
 \tan \frac{A}{2} &= \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} \\
 &= \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}} \\
 \therefore \tan \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\
 \text{Similarly } \tan \frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} \\
 \text{and } \tan \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}
 \end{aligned} \quad . \quad . \quad . \quad (\text{vii})$$

Note. Since the relations (v), (vi), are true for any triangle, the relations (vii) are true for any triangle.

§ 39. To prove that in any triangle $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$

$$\sin B = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\text{and } \sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)}.$$

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$$

$$= 2 \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$$

$$\left. \begin{array}{l} \text{Similarly } \sin B = \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)} \\ \text{and } \sin C = \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)} \end{array} \right\} \quad (\text{viii})$$

Note. Since the relations (v), (vi), are true for any triangle, the relations (viii) are true for any triangle.

§ 40. To prove that in any triangle $\Delta = \frac{1}{2}bc \sin A$
 $= \frac{1}{2}ca \sin B$
 $= \frac{1}{2}ab \sin C.$

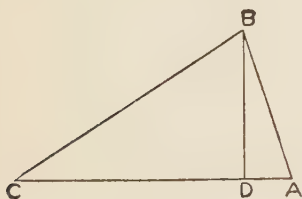


FIG. 64

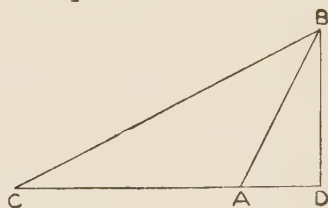


FIG. 65

Let BD (Figs. 64, 65), perpendicular to CA, meet CA or CA produced in D. In Fig. 64, $\hat{A}$ is acute; in Fig. 65, $\hat{A}$ is obtuse.

$$\begin{aligned} \text{In Fig. 64, } \Delta &= \frac{1}{2} CA \cdot DB \\ &= \frac{1}{2} b \cdot c \sin A \end{aligned}$$

$$\begin{aligned} \text{In Fig. 65, } \Delta &= \frac{1}{2} CA \cdot DB \\ &= \frac{1}{2} b \cdot c \sin \hat{DAB} \end{aligned}$$

$$= \frac{1}{2} b \cdot c \sin A \quad (\hat{DAB} = 180^\circ - A).$$

$$\therefore \text{ in both cases } \Delta = \frac{1}{2}bc \sin A \left\{ \begin{array}{l} \text{Similarly } \Delta = \frac{1}{2}ca \sin B \\ \text{and } \Delta = \frac{1}{2}ab \sin C \end{array} \right\} \quad . \quad . \quad . \quad (\text{ix})$$

Note. If $A = 90^\circ$, $\Delta = \frac{1}{2}bc$
 $= \frac{1}{2}bc \sin A \quad (\sin 90^\circ = 1).$

Hence the relations (ix) are true for any triangle.

§ 41. To prove that in any triangle $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

$$\begin{aligned} \Delta &= \frac{1}{2}bc \sin A \\ &= bc \sin \frac{A}{2} \cos \frac{A}{2} \\ &= bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}} \\ &= \sqrt{s(s-a)(s-b)(s-c)} \quad . \quad . \quad . \quad (\text{x}) \end{aligned}$$

Note. Since the relations (v), (vi), are true for any triangle, the relation (x) is true for any triangle.

§ 42. To prove that in any triangle $\frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}} = \frac{a-b}{a+b}$

$$\frac{\tan \frac{B-C}{2}}{\tan \frac{B+C}{2}} = \frac{b-c}{b+c}$$

$$\text{and } \frac{\tan \frac{C-A}{2}}{\tan \frac{C+A}{2}} = \frac{c-a}{c+a}.$$

$$\begin{aligned} \frac{a-b}{a+b} &= \frac{2R \sin A - 2R \sin B}{2R \sin A + 2R \sin B} \\ &= \frac{\sin A - \sin B}{\sin A + \sin B} \\ &= \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}} \\ &= \frac{\tan \frac{A-B}{2}}{\tan \frac{A+B}{2}} \end{aligned}$$

EXERCISES

Prove that in any triangle

$$1. \Delta = bc \sin \frac{A}{2} \cos \frac{A}{2}.$$

$$2. \Delta = Ra \sin B \sin C.$$

$$3. \Delta = \frac{b^2 \sin C \sin A}{2 \sin B}.$$

$$4. b \sin^2 \frac{C}{2} + c \sin^2 \frac{B}{2} = s - a.$$

$$5. b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} = s.$$

$$6. b \left(a \cos^2 \frac{C}{2} - c \cos^2 \frac{A}{2} \right) = s(a - c).$$

$$7. c \sin \frac{A}{2} \sin \frac{B}{2} = (s - c) \sin \frac{C}{2}.$$

$$8. a \cos \frac{B}{2} \cos \frac{C}{2} = s \sin \frac{A}{2}.$$

$$9. \tan \frac{A}{2} \cot \frac{B}{2} = \frac{c + a - b}{c - a + b}.$$

$$10. (s - b) \sin B \cos^2 \frac{C}{2} = (s - c) \sin C \cos^2 \frac{B}{2}.$$

$$11. s \left(1 - \tan \frac{A}{2} \tan \frac{B}{2} \right) = c.$$

$$12. \Delta = s^2 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}.$$

$$13. a \sin \frac{B}{2} \sin \frac{C}{2} = \sin \frac{A}{2} \left(b \sin^2 \frac{C}{2} + c \sin^2 \frac{B}{2} \right).$$

$$14. s = 4R \cos \frac{A}{2} \left(\sin \frac{A}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \right).$$

§ 43. A careful reading of the following examples will assist the student in answering the questions of the Exercises on page 112.

Example 1. Prove that in any triangle

$$a \cos A + b \cos B \cos 2C + c \cos C \cos 2B = 0.$$

$$\begin{aligned} & a \cos A + b \cos B \cos 2C + c \cos C \cos 2B \\ &= 2R \sin A \cos A + 2R \sin B \cos B \cos 2C + 2R \sin C \cos C \cos 2B \end{aligned}$$

$$\begin{aligned}
 &= R (\sin 2A + \sin 2B \cos 2C + \sin 2C \cos 2B) \\
 &= R (\sin 2A + \sin \overline{2B + 2C}) \\
 &= R (\sin 2A - \sin 2A) \\
 &= 0.
 \end{aligned}$$

Example 2. Prove that in any triangle

$$\begin{aligned}
 (b-c) \cos \frac{A}{2} &= a \sin \frac{B-C}{2}. \\
 a \sin \frac{B-C}{2} &= a \left(\sin \frac{B}{2} \cos \frac{C}{2} - \cos \frac{B}{2} \sin \frac{C}{2} \right) \\
 &= a \left(\sqrt{\frac{(s-c)(s-a)}{ca}} \cdot \frac{s(s-c)}{ab} - \sqrt{\frac{s(s-b)}{ca}} \cdot \frac{(s-a)(s-b)}{ab} \right) \\
 &= \sqrt{\frac{s(s-a)}{bc}} \left\{ (s-c) - (s-b) \right\} \\
 &= (b-c) \cos \frac{A}{2}.
 \end{aligned}$$

Example 3. Prove that in any triangle

$$\Sigma a \cos A \cos 2A + 2a \cos A \sin 2B \sin 2C = 0.$$

$$\Sigma a \cos A \cos 2A = \Sigma 2R \sin A \cos A \cos 2A$$

$$= R \Sigma \sin 2A \cos 2A$$

$$= \frac{R}{2} \Sigma \sin 4A$$

$$= \frac{R}{2} (\sin 4A + \sin 4B + \sin 4C)$$

$$= R (\sin \overline{2A + 2B} \cos \overline{2A - 2B} + \sin 2C \cos 2C)$$

$$= -R \sin 2C (\cos \overline{2A - 2B} - \cos 2C)$$

$$= -R \sin 2C (\cos \overline{2A - 2B} - \cos \overline{2A + 2B})$$

$$= -2R \sin 2C \sin 2A \sin 2B$$

$$= -4R \sin A \cos A \sin 2B \sin 2C$$

$$= -2a \cos A \sin 2B \sin 2C$$

$$\therefore \Sigma a \cos A \cos 2A + 2a \cos A \sin 2B \sin 2C = 0.$$

Note. $\Sigma a \cos A \cos 2A$ means the sum of all terms of the type $a \cos A \cos 2A$, i.e., $a \cos A \cos 2A + b \cos B \cos 2B + c \cos C \cos 2C$; in the same way $\Sigma \sin 4A$ means $\sin 4A + \sin 4B + \sin 4C$.

Example 4. Prove that the triangle **ABC** is isosceles if $\sin \frac{A}{2} = \frac{a}{b+c}$.

$$\begin{aligned}\frac{a}{b+c} &= \frac{\sin A}{\sin B + \sin C} \\ &= \frac{2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \sin \frac{B+C}{2} \cos \frac{B-C}{2}} \\ &= \frac{\sin \frac{A}{2}}{\cos \frac{B-C}{2}} \\ \therefore \cos \frac{B-C}{2} &= 1\end{aligned}$$

$$\therefore B = C$$

$\therefore$ the triangle **ABC** is isosceles.

EXERCISES

Prove that in any triangle

- $(a - b \sin A)(1 + \sin B) = a \cos^2 B$.
- $a \cos A \cos 2A \sin 4B = b \cos B \cos 2B \sin 4A$.
- $a \cos A (\sin 2B + \sin 2C) = \sin 2A (b \cos B + c \cos C)$.
- $abc \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = \Delta s$.
- $c^2 \sin (A - B) = (a^2 - b^2) \sin C$.
- $4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = s - a$.
- $\Delta^2 = abc s \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.
- $c \left(\sin^2 \frac{A}{2} - \sin^2 \frac{B}{2} \right) = (a - b) \cos^2 \frac{C}{2}$.
- $\tan (A - B) = \frac{b \cos B - a \cos A}{b \cos B + a \cos A} \tan C$.
- $a \cos A \sin (B - C) + \sin A (b \cos B - c \cos C) = 0$.
- $a \operatorname{cosec} \frac{A}{2} \sin \frac{B-C}{4} = \cos \frac{\pi-A}{4} \left(c \operatorname{cosec} \frac{C}{2} - b \operatorname{cosec} \frac{B}{2} \right)$.
- $a \operatorname{cosec} \frac{A}{2} \cos \frac{B-C}{4} = \sin \frac{\pi-A}{4} \left(b \operatorname{cosec} \frac{B}{2} + c \operatorname{cosec} \frac{C}{2} \right)$.
- $a \cos A + b \cos B - c \cos C = 2c \cos A \cos B$.
- $a \cos A \cos 2A + b \cos B \cos 2B \cos 4C + c \cos C \cos 2C \cos 4B = 0$.

15. $(b \cos B \cos 2B - c \cos C \cos 2C) \sin 2A = a \cos A \cos 2A \sin 2(B - C).$
16. $a \sin A + b \sin B - c \sin C = 4R \sin A \sin B \cos C.$
17. $a \cos^2 \frac{A}{2} + b \cos^2 \frac{B}{2} - c \cos^2 \frac{C}{2} - 2R \cos A \cos B \sin C = s - c.$
18. $a \sin^2 \frac{A}{2} + b \sin^2 \frac{B}{2} - c \sin^2 \frac{C}{2} + 2R \cos A \cos B \sin C = s - c.$
19. $a \cot A + b \cot B - c \cot C = 2R \left(4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} - 1 \right).$
20. $(b + c) \sin \frac{A}{2} = a \cos \frac{B - C}{2}.$
21. $\frac{\cos 2A}{a \cos A} + \frac{\cos 2B}{b \cos B} + \frac{c \cos C}{ab \cos A \cos B} = 0.$
22. $(a^2 \cos^2 A - b^2 \cos^2 B) \sin 2C + c^2 \cos^2 C \sin 2(A - B) = 0.$
23. $a^2 \operatorname{cosec}^2 \frac{A}{2} + b^2 \operatorname{cosec}^2 \frac{B}{2} - c^2 \operatorname{cosec}^2 \frac{C}{2} = 2ab \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \sin \frac{C}{2}.$
24. $\cos 2B (b \cos B + c \cos C \cos 2A) = \cos 2C (c \cos C + b \cos B \cos 2A).$
25. $c^2 \cos^2 C = a^2 \cos^2 A + b^2 \cos^2 B + 2ab \cos A \cos B \cos 2C.$
26. $(a \cos A - b \cos B)^2 = c^2 \cos^2 C - 4ab \cos A \cos B \cos^2 C.$
27. $(b \cos B + a \cos A \cos 2C) (c \cos C + a \cos A \cos 2B)$
 $= bc \cos B \cos C \cos^2 2A.$
28. $\Sigma a \cos A = 2a \sin B \sin C.$
29. $\Sigma a \cot A = 2R \left(1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right).$
30. $\Sigma a \sin^2 \frac{A}{2} = s - 2R \sin A \sin B \sin C.$
31. $\Sigma a \cos^2 \frac{A}{2} = s + 2R \sin A \sin B \sin C.$
32. $\Sigma a \cos (B - C) = 8R \sin A \sin B \sin C.$
33. $\Sigma (b - c) \sin 2A = 2 \Sigma a \cos A (\sin B - \sin C).$
34. $\Sigma a \cos A (\sin 2B - \sin 2C) = 0.$
35. $\Sigma \frac{b - c}{a} \cos^2 \frac{A}{2} = 0.$
36. $\Sigma a \sin (B - C) = 0.$
37. $\Sigma a^2 \cos^2 A + 2 \Sigma bc \cos B \cos C \cos 2A = 0.$
38. $\Sigma a \cos A + \Sigma \cos 2A (b \cos B + c \cos C) = 0.$

Prove the following :

39. If $\frac{b - c}{a} \cos^2 \frac{A}{2} + \frac{c - a}{b} \cos^2 \frac{B}{2} = 0$, the triangle ABC is isosceles.
40. If $a \sin (B - C) = b \sin (A - C)$, the triangle ABC is isosceles.

41. If $b + c = \sqrt{2}a \cos \frac{B - C}{2}$, the triangle **ABC** is right-angled.
42. If $b \left(1 - \tan \frac{B - C}{2}\right) = c \left(1 + \tan \frac{B - C}{2}\right)$, the triangle **ABC** is right-angled.
43. If $a \sin A + b \sin B = c \sin C$, the triangle **ABC** is right-angled.

CHAPTER XI

LOGARITHMS

§ 44. The study of logarithms is essentially algebraic, and we shall give only a brief summary of the laws involved in the use of tables of logarithms.

DEFINITION. The logarithm of a number to a given base is the index of that power of the base which is equal to the given number.

e.g., if $a^m = x$, then $\log_a x = m$.

LAWS OF LOGARITHMS

Law I. The logarithm of the product of two or more numbers is equal to the sum of the logarithms of the numbers.

$$\text{i.e., } \log_a xy = \log_a x + \log_a y.$$

Law II. The logarithm of the quotient of two numbers is equal to the logarithm of the dividend minus the logarithm of the divisor.

$$\text{i.e., } \log_a \frac{x}{y} = \log_a x - \log_a y.$$

Law III. The logarithm of any power of a number is equal to the index of the power multiplied by the logarithm of the number.

$$\text{i.e., } \log_a x^m = m \log_a x.$$

Change of Base. The logarithms of any number x to two bases a and b are related thus :

$$\log_b x = \log_a x \div \log_a b.$$

We shall use only common logarithms, *i.e.*, logarithms to the base 10 : the necessary tables are given on pp. 174-75. When no base is mentioned, it is understood that a logarithm is a common logarithm.

§ 45. LOGARITHMS OF THE CIRCULAR FUNCTIONS

In § 13 we explained how to use tables of natural sines, cosines, and tangents. It is obviously convenient to read off directly the logarithms of the circular functions, and these are given in tables which are used in a similar manner to those of the natural sines, etc.

e.g., from table of natural sines, $\sin 61^\circ 38' = .8799$
and from table of logarithms of sines, $\log \sin 61^\circ 38' = \bar{1}.9444$.

Note. Very often the values of the logarithms of the circular functions are increased by 10, and the tables are headed "Tabular Logarithm of Sine," etc., or "Logarithmic Sine," etc.

e.g., $\log \sin 61^\circ 38' = \bar{1}.9444$
and logarithmic $\sin 61^\circ 38' = \bar{1}.9444 + 10 = 9.9444$
or, in contracted form, $L \sin 61^\circ 38' = 9.9444$.

Example 1. Evaluate $\frac{412.5 \sin 74^\circ 26'}{\sin 229^\circ 33'}$.

Nos.	Logs.
412.5	2.6154
$\sin 74^\circ 26'$	$\bar{1}.9838$
	2.5992 = sum
$-\sin 229^\circ 33' = \sin 49^\circ 33'$	$\bar{1}.8813$
522.3	2.7179 = difference

$$\therefore \frac{412.5 \sin 74^\circ 26'}{\sin 229^\circ 33'} = -522.$$

Note 1. The logarithm of the numerical value of the given expression $= \log 412.5 + \log \sin 74^\circ 26' - \log \sin 49^\circ 33'$.

Note 2. The sign of the expression is determined independently of the logarithmic calculation.

Example 2. Prove that

$$\begin{aligned} \log (\sin A - \cos A) &= \frac{1}{2} \log 2 + \log \sin (A - 45^\circ). \\ \frac{1}{2} \log 2 + \log \sin (A - 45^\circ) &= \log \sqrt{2} + \log \sin (A - 45^\circ) \\ &= \log \{ \sqrt{2} \sin (A - 45^\circ) \} \\ &= \log \{ \sqrt{2} (\sin A \cos 45^\circ - \cos A \sin 45^\circ) \} \\ &= \log (\sin A - \cos A). \end{aligned}$$

EXERCISES

Evaluate

1. $38.52 \times \sin 42^\circ 34'$.
2. $4.278 \times \tan 59^\circ 15'$.
3. $8.13 \times \operatorname{cosec} 61^\circ 44'$.
4. $5.46 \times \sec 124^\circ 25'$.
5. $\sin 63^\circ 25' \cos 63^\circ 25'$.
6. $\tan 23^\circ 46' \cot 70^\circ 32'$.
7. $\frac{14.45 \sin 32^\circ 55'}{25.63}$.
8. $\frac{31.02 \sin 57^\circ 34'}{\sin 76^\circ 19'}$.
9. $\frac{4.71 \tan 42^\circ 47'}{29.47}$.
10. $\frac{469 \sec 100^\circ 51'}{\cot 132^\circ 39'}$.
11. $\left(\frac{\sin 63^\circ 24'}{\cos 59^\circ 16'} \right)^{1.52}$.
12. $\left(\frac{\sin 31^\circ 7'}{\cos 81^\circ 17'} \right)^{2.05}$.

Prove that

13. $\log (\sin^2 A + \cos^2 A) = 0$.
14. $\log \cos A + \log \tan A = \log \sin A$.
15. $\log \sin A + \log \cot A = \log \cos A$.
16. $\log \cos 2A = \log (\cos A + \sin A) + \log (\cos A - \sin A)$.
17. $\log \tan 2A = \log 2 + \log \tan A - \log (1 + \tan A) - \log (1 - \tan A)$.
18. $\log \sin 3A = \log \sin A + \log (\sqrt{3} + 2 \sin A) + \log (\sqrt{3} - 2 \sin A)$.
19. $\log (\sin A + \cos A) = \frac{1}{2} \log 2 + \log \sin (A + 45^\circ)$.
20. $\log (1 + \cos A) + \log (1 - \cos A) = \log \cos A + \log \sin A + \log \tan A$.
21. $\log \tan A - 2 \log \sec A = \log \sin A + \log \cos A$.
22. $\log \left(\sqrt{2} + \sec \frac{A}{2} \right) + \log \left(\sqrt{2} - \sec \frac{A}{2} \right) - 2 \log \sec \frac{A}{2} = \log \cos A$.
23. $\frac{\log \sin^4 A}{\log (1 + \cos A) + \log (1 - \cos A)} = 2$.
24. $\log \sin 8\theta - \log \sin 2\theta = 2 \log 2 + \log \cos 2\theta + \log \cos 4\theta$.
25. Evaluate (a) $10 \log \sin 45^\circ$, (b) $e^{2 \log_e \tan \frac{2\pi}{3}}$.
26. $a = 2R \sin A$. Find R if $a = 15.02$ and $A = 134^\circ 16'$.
27. $\mu = \frac{\sin i}{\sin r}$. Find the least positive value of r if $\mu = 1.52$ and $i = 31^\circ 44'$.
28. $x = \frac{a(\cos \beta - \cos \alpha)}{\cos \alpha + \cos \beta}$. Find x if $a = 105.5$, $\alpha = 51^\circ 40'$, and $\beta = 32^\circ 15'$.
29. $y = \frac{2a \sin \alpha \sin \beta}{\sin \alpha - \sin \beta}$. Find y if $a = 325$, $\alpha = 65^\circ 25'$, and $\beta = 19^\circ 29'$.

In the following exercises A , B , and C are angles of a triangle.

30. $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$. Find r if $R = 2.423$, $A = 49^\circ 35'$, and $B = 62^\circ 34'$.

31. $s = 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$. Find s if $R = 31.4$, $A = 47^\circ 12'$, and $B = 31^\circ 42'$.
32. $\sin B = \frac{12.92 \sin A}{11.25}$. Find the values of B if $A = 49^\circ 47'$.
33. $\sin A = \frac{485 \sin C}{721}$. Find the value of A if $C = 69^\circ 21'$.
34. $\cot \frac{C}{2} = \frac{a+b}{a-b} \tan \frac{A-B}{2}$. Find C if $a = 17.32$, $b = 12.15$, and $A - B = 17^\circ 41'$.
35. $c \cos \frac{A-B}{2} = (a+b) \sin \frac{C}{2}$. Calculate the values of the angles of the triangle in which $c = 28.5$, $a + b = 41.7$, $A - B = 19^\circ 21'$.

CHAPTER XII

SOLUTION OF TRIANGLES

§ 46. The right-angled triangle was discussed in Chapter VI. We now proceed to consider how to solve any triangle. The following four cases arise, the first three corresponding to the elementary geometrical theorems on congruent triangles.

- A. Given two angles and one side.
- B. Given three sides.
- C. Given two sides and the included angle.
- D. Given two sides and the angle opposite to one of them.

§ 47. *Case A.* Given two angles and one side.

e.g., given $A = 78^\circ 30'$, $B = 46^\circ 45'$, $a = 12.21$.

$$\text{Use the formula } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

which may be written $a = 2R \sin A$, $b = 2R \sin B$, $c = 2R \sin C$

$$\begin{aligned} C &= 180^\circ - (A + B) \\ &= 180^\circ - 125^\circ 15' \\ &= 54^\circ 45' \end{aligned}$$

Formulae	Numbers	Logarithms
$2R = \frac{a}{\sin A}$	$a = 12.21$	1.0867
	$\sin A = \sin 78^\circ 30'$	$\bar{1}.9912$
	$2R$	$1.0955 = \text{difference}$
$b = 2R \sin B$	$2R$	1.0955
	$\sin B = \sin 46^\circ 45'$	$\bar{1}.8624$
	$\therefore b = 9.076$	$0.9579 = \text{sum}$
$c = 2R \sin C$	$2R$	1.0955
	$\sin C = \sin 54^\circ 45'$	$\bar{1}.9121$
	$\therefore c = 10.17$	$1.0076 = \text{sum}$

Results $C = 54^\circ 45'$ $b = 9.08$ $c = 10.2$.

Note 1. $\log 2R = \log a - \log \sin A$.

The value of $2R$ is not required.

Note 2. $\log b = \log 2R + \log \sin B$.

Note 3. b and c may be found directly from the formulæ

$$b = \frac{a \sin B}{\sin A}, \quad c = \frac{a \sin C}{\sin A}.$$

EXERCISES

Solve the $\triangle ABC$ in the following cases :

1. $A = 71^\circ 40'$, $B = 41^\circ 50'$, $c = 30.25$.
2. $C = 38^\circ 49'$, $A = 79^\circ 26'$, $a = 29.02$.
3. $B = 82^\circ 9'$, $C = 65^\circ 25'$, $c = 21.85$.
4. $C = 114^\circ 36'$, $A = 40^\circ 6'$, $b = 8.351$.
5. $B = 40^\circ 9'$, $C = 99^\circ 31'$, $a = 526.1$.
6. $A = 62^\circ 20'$, $B = 52^\circ 50'$, $b = 1544$.
7. $B = 100^\circ 15'$, $C = 58^\circ 53'$, $c = 123.4$.
8. $C = 55^\circ 50'$, $B = 79^\circ 12'$, $a = 29.62$.
9. $B = 26^\circ 47'$, $A = 124^\circ 24'$, $c = 89.35$.
10. $C = 46^\circ 50'$, $A = 19^\circ 59'$, $b = 139.8$.

§ 48. *Case B.* Given three sides.

First Method; e.g., given $a = 472.6$, $b = 670.1$, $c = 854.7$.

Use the formula $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

$$a = 472.6 \quad \therefore s = 998.7$$

$$b = 670.1 \quad s - a = 526.1$$

$$c = 854.7 \quad s - b = 328.6$$

$$\therefore 2s = 1997.4 \quad s - c = 144.0$$

Formulæ	Numbers	Logarithms
$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$	$s - b = 328.6$	2.5167
	$s - c = 144.0$	2.1584
	$s = 998.7$	2.9994
	$s - a = 526.1$	2.7211
		sum = 4.6751
		sum = 5.7205
		difference = 2.9546
		difference $\div 2$, = 1.4773
	$\frac{A}{2} = 16^\circ 42'$	
	$\therefore A = 33^\circ 24'$	

Formulae	Numbers	Logarithms
$\tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$	$s - c = 144.0$	2.1584
	$s - a = 526.1$	2.7211 sum = 4.8795
	$s = 998.7$	2.9994
	$s - b = 328.6$	2.5167 sum = 5.5161
		difference = 1.3634
	$\therefore \frac{B}{2} = 25^\circ 40'$ $\therefore B = 51^\circ 20'$	difference $\div 2$, = 1.6817
$\tan \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$	$s - a = 526.1$	2.7211
	$s - b = 328.6$	2.5167 sum = 5.2378
	$s = 998.7$	2.9994
	$s - c = 144.0$	2.1584 sum = 5.1578
		difference = 0.0800
	$\therefore \frac{C}{2} = 47^\circ 38'$ $\therefore C = 95^\circ 16'$	difference $\div 2$, = 0.0400

Results $A = 33^\circ 24'$
 $B = 51^\circ 20'$
 $C = 95^\circ 16'.$

Note 1. When only one angle is required, the formulæ for the sine or cosine of the half-angle are quite suitable. When all three angles have to be found, the tangent formula is preferable, for we then use only 4 logs, viz., those of s , $s - a$, $s - b$, and $s - c$, and these enter into the calculation of all three angles.

Note 2. When two angles have been determined, the third can be found from the relation $A + B + C = 180^\circ$. The student is advised, however, to find the three angles separately, and check his results by testing whether their sum is approximately 180° . Since the logarithms employed are approximate numbers, the sum of the three angles obtained may differ from 180° by a few minutes.

Second Method ; e.g., given $a = 16.21$, $b = 19.05$, $c = 14.37$.

$$\begin{aligned}\tan \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ &= \sqrt{\frac{(s-a)(s-b)(s-c)}{s} \div (s-a)}\end{aligned}$$

$a = 16.21$	$s = 24.82$ $s - a = 8.61$	$\log s = 1.3949$ $\log (s - a) = 0.9350$		$\log \tan \frac{A}{2} = \bar{1}.7252$ $\frac{A}{2} = 27^{\circ} 59'$	$A = 55^{\circ} 58'$
$b = 19.05$	$s - b = 5.77$	$\log (s - b) = 0.7612$	$\log \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = 0.6602$	$\log \tan \frac{B}{2} = \bar{1}.8990$ $\frac{B}{2} = 38^{\circ} 24'$	$B = 76^{\circ} 48'$
$c = 14.37$	$s - c = 10.45$	$\log (s - c) = 1.0191$		$\log \tan \frac{C}{2} = \bar{1}.6411$ $\frac{C}{2} = 23^{\circ} 38'$	$C = 47^{\circ} 16'$
$\therefore 2s = 49.63$					

Note 1. $\log \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$

$$= \frac{1}{2} \{ \log (s-a) + \log (s-b) + \log (s-c) - \log s \}.$$

$$\text{Note 2. } \log \tan \frac{A}{2} = \log \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} - \log (s-a)$$

$$= 0.6602 - 0.9350 = \bar{1}.7252$$

and similarly for $\log \tan \frac{B}{2}$, $\log \tan \frac{C}{2}$.

EXERCISES

Solve the ΔABC in the following cases :

1. $a = 11.25$, $b = 21.43$, $c = 28.72$.
2. $a = 613.2$, $b = 430.8$, $c = 331.7$.
3. $a = 1997$, $b = 3621$, $c = 2803$.
4. $a = 130.8$, $b = 79.36$, $c = 154.1$.
5. $a = 15.27$, $b = 8.16$, $c = 10.39$.
6. $a = 1042$, $b = 1874$, $c = 2573$.
7. $a = 201.3$, $b = 376.2$, $c = 301.8$.
8. $a = 10.69$, $b = 13.15$, $c = 9.66$.
9. $a = 7.29$, $b = 11.61$, $c = 8.25$.
10. $a : b : c = 54.5 : 71.7 : 98.9$.

§ 49. *Case C.* Given two sides and the included angle.
e.g., given $a = 103.5$, $b = 43.21$, and $C = 84^\circ 39'$.

Use the formula $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \tan \frac{A+B}{2}$

$$a = 103.5$$

$$A+B = 180^\circ - 84^\circ 39'$$

$$b = 43.21$$

$$= 95^\circ 21'$$

$$\therefore a-b = 60.29$$

$$\therefore \frac{A+B}{2} = 47^\circ 41' \text{ (to nearest minute)}$$

$$\text{and } a+b = 146.71$$

Formulae	Numbers	Logarithms
$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \tan \frac{A+B}{2}$	$a-b = 60.29$	1.7802
	$a+b = 146.7$	2.1665
		$\bar{1}.6137 = \text{difference}$
	$\tan \frac{A+B}{2} = \tan 47^\circ 41'$	0.0408
	$\therefore \frac{A-B}{2} = 24^\circ 17'$	$\bar{1}.6545 = \text{sum}$
$c = \frac{a \sin C}{\sin A}$	$\therefore A = 71^\circ 58'$	
	and $B = 23^\circ 24'$	
	$a = 103.5$	2.0149
	$\sin A = \sin 71^\circ 58'$	$\bar{1}.9782$
		$2.0367 = \text{difference}$
	$\sin C = \sin 84^\circ 39'$	$\bar{1}.9981$
	$\therefore c = 108.3$	$2.0348 = \text{sum}$

$$\text{Results } A = 71^\circ 58'$$

$$B = 23^\circ 24'$$

$$c = 108.$$

$$\text{Note. } \log \tan \frac{A-B}{2} = \log (a-b) - \log (a+b) + \log \tan \frac{A+B}{2}.$$

EXERCISES

Solve the ΔABC in the following cases :

1. $a = 17.25$, $b = 10.19$, $C = 53^\circ 25'$.
2. $b = 29.87$, $c = 32.71$, $A = 63^\circ 38'$.
3. $a = 30.56$, $b = 23.07$, $C = 112^\circ 25'$.
4. $c = 18.04$, $a = 14.41$, $B = 67^\circ 43'$.
5. $b = 6086$, $a = 4994$, $C = 109^\circ 29'$.
6. $b = 309.4$, $c = 257.1$, $A = 79^\circ 37'$.
7. $a = 10.14$, $c = 16.92$, $B = 52^\circ 8'$.
8. $c = 5821$, $b = 6949$, $A = 25^\circ 25'$.
9. $a = 708.2$, $c = 996.6$, $B = 106^\circ 49'$.
10. $b = 19.85$, $a = 28.88$, $C = 34^\circ 21'$.

§ 50. *Case D.* Given two sides and the angle opposite to one of them.
e.g., given b , c , C .

$$\text{Use the formula } \sin B = \frac{b \sin C}{c}.$$

Since $\frac{b \sin C}{c}$ gives the value of $\sin B$, we have still to decide whether B is acute or obtuse.

(a) If $C \geq 90^\circ$, then B must be acute.

(b) If $C < 90^\circ$ and (i) $c \geq b$, then B must be acute.

(ii) $c < b \sin C$, then $\sin B > 1$, which is impossible.

(iii) $c = b \sin C$, then $\sin B = 1$, $\therefore B = 90^\circ$.

(iv) $c > b \sin C$ and $c < b$, then B may be acute or obtuse.

From the above deductions we learn that two triangles satisfy the given data if C is acute and c is less than b but greater than $b \sin C$. Since two solutions are possible, the case is referred to as the *Ambiguous Case*.

The results which we have obtained analytically may be easily verified geometrically.

Let $\hat{XCY}$ (Figs. 66, 67) be the acute angle C .

From CY cut off $CA = b$, and draw AD perpendicular to CX .

Then $AD = b \sin C$.

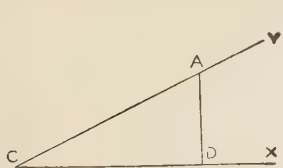


FIG. 66

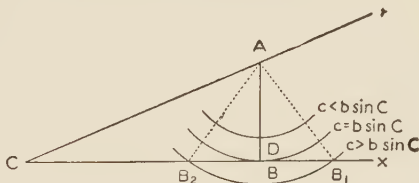


FIG. 67

With centre A (Fig. 67) and radius c describe an arc ; then the drawing illustrates what happens in cases (ii), (iii), (iv), of subsection (b) above, for

in (ii) if $c < b \sin C$ the arc does not meet CX , and no triangle is formed ;

in (iii) if $c = b \sin C$ the arc touches CX at D , and ABC is a right-angled triangle ;

and in (iv) if $c > b \sin C$ and $< b$ the arc cuts CX in two points B_1, B_2 , and two triangles AB_1C, AB_2C are formed. The two values of A may be denoted by A_1, A_2 , and the corresponding values of a by a_1, a_2 .

Note. The Ambiguous Case may be demonstrated algebraically. Since b, c , and C are given, then a may be found from the formula

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\text{i.e., } a^2 - 2ab \cos C + b^2 - c^2 = 0$$

which may be looked upon as a quadratic equation in which the unknown is a .

$$\begin{aligned} \text{The equation gives } a &= b \cos C \pm \sqrt{b^2 \cos^2 C - (b^2 - c^2)} \\ &= b \cos C \pm \sqrt{c^2 - b^2 \sin^2 C}. \end{aligned}$$

Hence

(i) If $c > b$, the values of a are real, but one is negative.

If $c = b$, one value of a is real and positive, the other is zero.

(ii) If $c < b \sin C$, the values of a are imaginary.

(iii) If $c = b \sin C$, the values of a are real, positive and equal.

(iv) If $c > b \sin C$ and $< b$, the values of a are real, unequal and positive.

e.g., given $b=157.4$, $c=144.2$, $C=48^\circ 35'$.

Use the formula $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$.

Formulæ	Numbers	Logarithms	Results
$2R = \frac{c}{\sin C}$	$c = 144.2$	2.1590	$C = 48^\circ 35'$
	$\sin C = \sin 48^\circ 35'$	$\bar{1}.8751$	$B_1 = 54^\circ 57'$
	2R	2.2839 = difference	$B_2 = 125^\circ 3'$
$\sin B = \frac{b}{2R}$	$b = 157.4$	2.1970	$A_1 = 76^\circ 28'$
	2R	2.2839	$A_2 = 6^\circ 22'$
	$\sin B$	$\bar{1}.9131$ = difference	
$a_1 = 2R \sin A_1$	2R	2.2839	$a_1 = 187$
	$\sin A_1 = \sin 76^\circ 28'$	$\bar{1}.9877$	
		2.2716 = sum	
$a_2 = 2R \sin A_2$	2R	2.2839	$a_2 = 21.8$
	$\sin A_2 = \sin 6^\circ 22'$	$\bar{1}.0447$	
		1.3286 = sum	

Note 1. From the value of $\log \sin B$ we obtain the acute angle B_1 and its supplement B_2 .

Note 2. $A_1 = 180^\circ - (B_1 + C)$, $A_2 = 180^\circ - (B_2 + C)$; $a_1 = CB_1$, $a_2 = CB_2$.

Note 3. It is unnecessary to determine the number of solutions before proceeding to solve the triangle. If, for example, there had been no solution in the preceding example, a positive value for $\log \sin B$ would have been obtained: such a value is inadmissible. If there had been only one solution then either B_1 would have been found to be 90° , in which case B_2 would have been 90° , or else there would have been obtained for A_2 a negative value, which is inadmissible.

EXERCISES

State the number of solutions in the following cases:

- $b=479.1$, $c=365.4$, $B=34^\circ 26'$.
- $c=268.4$, $a=371.9$, $C=29^\circ 40'$.

$$3. a = 11.25, \quad b = 25.25, \quad A = 31^\circ 30'.$$

$$4. a = 18.48, \quad b = 9.24, \quad B = 30^\circ.$$

Find the angles of the following triangles :

$$5. a = 11.84, \quad b = 10.86, \quad B = 54^\circ 36'.$$

$$6. b = 209.6, \quad c = 231.5, \quad C = 61^\circ 19'.$$

$$7. a = 27.95, \quad c = 29.75, \quad A = 39^\circ 45'.$$

Solve the $\triangle ABC$ in the following cases :

$$8. b = 15.16, \quad c = 23.55, \quad C = 131^\circ 40'.$$

$$9. a = 13.64, \quad b = 18.23, \quad A = 34^\circ 45'.$$

$$10. a = 171.5, \quad b = 139.2, \quad B = 41^\circ 35'.$$

EXERCISES

1. $a = 76.41$, $b = 50.28$, $c = 99.51$. Find the greatest angle and the area of the triangle.

2. $a = 11.79$, $b = 14.05$, $c = 8.38$. Find the radius of the circumcircle and the smallest angle of the triangle.

3. $a = 13.42$, $c = 10.84$, $B = 62^\circ 32'$. Find the area of the triangle and the length of the altitude drawn from the vertex C .

4. $B = 37^\circ 45'$, $C = 71^\circ 16'$, $a = 7.45$. Calculate the lengths of the smallest side and the radius of the circumcircle.

5. Find the least angle of the triangle whose sides are in the proportion of 8, 11, 14.

6. $a = 3.69$, $A = 49^\circ 35'$, $B = 62^\circ 34'$. Calculate the values of R and b .

7. $b = 125.5$, $c = 142.7$, $A = 55^\circ 45'$. Find the area of the triangle and the length of the circumradius.

8. $a = 31$, $b = 23$, $A = 51^\circ 32'$. Calculate the values of B and R .

9. In $\triangle ABC$, AD is the perpendicular from A to BC . If $AD = 9.65$ in., $\hat{DAC} = 26^\circ 30'$, and $\hat{DAB} = 38^\circ 10'$, calculate the lengths of the sides of $\triangle ABC$.

10. $a = 7.25$, $b = 9.16$, $c = 11.27$. Find the radius of the circumcircle, using the formula $abc = 4R \Delta$.

11. $a = 6.25$, $B = 39^\circ 30'$, $C = 90^\circ$. If CD is the median from C to the hypotenuse, find the length of the median and the area of $\triangle DBC$.

12. $\triangle ABC$ is isosceles: $A = 36^\circ$ and $a = 5.25$ in. The bisectors of the base angles B and C meet at O . Find the lengths of OB and the circumradius of $\triangle OBC$.

13. ABC is a triangle having $A = 64^\circ$, $B = 48^\circ$. The bisector of angle A meets BC in D . If $AD = 3.5$ in., calculate the lengths of the sides

of $\triangle ABC$, and check your results by constructing the triangle and measuring the sides.

14. $\triangle ABC$ is inscribed in a circle. Tangents are drawn to the circle at A , B , and C , forming $\triangle XYZ$. If $AB = 4.5$ in., $\hat{BAC} = 50^\circ$, and $\hat{CBA} = 60^\circ$ calculate the lengths of the sides of $\triangle XYZ$.

15. $a = 41.8$, $b = 31.6$, $c = 50.5$. The internal and external bisectors of angle A meet BC in X and Y respectively. Calculate the lengths of AX and AY .

16. If, given a , b , and B , two values for c can be found, prove (i) that the product of these values is $a^2 - b^2$, (ii) that their sum is $2a \cos B$.

17. Two adjacent sides of a triangular field measure 8.5 chains and 11.2 chains respectively, and contain an angle of $82^\circ 15'$. Find the perimeter and area of the field.

18. The longest side of a triangular field is 14.6 chains, and the angles subtended by the other two sides at the ends of this side are $34^\circ 12'$ and $57^\circ 44'$ respectively. Find the length of fencing required to surround the field.

19. XYZ is a triangular plot. $XY = 29.2$ yds., $YZ = 21.5$ yds., and $\hat{XZY} = 59^\circ 35'$. Find the other angles of the triangle and the area of the plot.

20. P and Q are two places such that the distance between them cannot be measured directly. If the distances of P and Q from a third place R are 9.5 and 10.5 miles respectively, and $\hat{PRQ} = 79^\circ 32'$, find the distance from P to Q and the inclination of PR to PQ .

21. Two trees on the bank of a stream are 17.5 yds. apart. A post on the opposite bank is between the trees, and the directions from the trees to the post make angles of $52^\circ 30'$ and $61^\circ 45'$ with the line joining the trees. Find the width of the stream.

22. Two straight roads from a point A are inclined to each other at an angle of 52° . A cyclist goes from A along one road at the rate of 12 miles per hour, and a motorist starts from A at the same time and proceeds along the other road. Find the speed of the latter if at the end of 90 minutes he is 29 miles distant from the cyclist.

23. $ABCD$ is a piece of ground in the form of a trapezium of which AB and DC are the parallel sides. If $AB = BD = 27$ chains, $DC = 17$ chains, and angle $BAD = 54^\circ 9'$ find the lengths of the other two sides.

24. A field $PQRS$ is right-angled at R ; $RS = 25$ chains, $PQ = 35$ chains, $PS = 42$ chains, and $\hat{QPS} = 41^\circ$. Find the length of QR .

25. At two points A and B on horizontal ground vertical posts AD

and **BC** are placed. **AD** is 6 ft. high and subtends an angle of $24^{\circ} 12'$ at **B**. The angle of elevation of **C** from **D** is $49^{\circ} 50'$. Find the length of **BC**.

26. On a level stretch of land, three straight roads **AE**, **AB**, and **AC** radiate from **A**. **AB** and **AC** are on the same side of **AE** and inclined to it at angles of $15^{\circ} 30'$ and $52^{\circ} 15'$ respectively, while the distance **AB** is 880 yds. At **B** one road leads to **C** and another, **BD**, is parallel to **AE**. If $\hat{CBD}$ is $82^{\circ} 36'$, find the distances of **C** from **A** and **B**.

27. At a signpost on a straight road a person observes two spires ahead of him on the road. He walks 300 yds. in a direction inclined at 80° to the road, and finds that the distances from the signpost to each spire subtend angles of $48^{\circ} 33'$ and $81^{\circ} 20'$ respectively at his new position. Find the distance between the spires.

CHAPTER XIII

PROBLEMS

§ 51. The problems in this chapter afford the student further opportunity of practice in the application of the relations connecting the sides and angles of triangles.

Example 1. At a point **A**, at which a vertical tower subtends an angle α , the angle of depression of the base **B** of the tower is β ; show that the height of the tower is $\frac{AB \sin \alpha}{\cos (\alpha - \beta)}$.

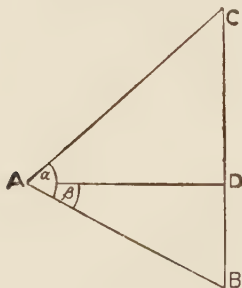


FIG. 68

Let **BC** (Fig. 68) represent the tower, and let **AD**, perpendicular to **BC**, meet **BC** in **D**.

$$\text{Then } \hat{CAD} = \alpha - \beta$$

$$\text{and } \therefore \hat{ACD} = 90^\circ - (\alpha - \beta)$$

$$\begin{aligned} \text{But } \frac{BC}{AB} &= \frac{\sin \hat{BAC}}{\sin \hat{ACB}} \\ &= \frac{\sin \alpha}{\sin \{90^\circ - (\alpha - \beta)\}} \end{aligned}$$

$$= \frac{\sin \alpha}{\cos (\alpha - \beta)}$$

$$\therefore BC = \frac{AB \sin \alpha}{\cos (\alpha - \beta)}.$$

Example 2. Two circles, whose radii differ by $2a$ and have product b^2 , have external contact; show that the tangent of the angle between their direct common tangents is $\pm \frac{2ab}{b^2 - a^2}$.

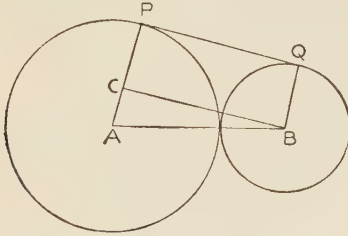


FIG. 69

Let PQ (Fig. 69) touch the circles, with centres A and B , at P and Q respectively. Draw BC parallel to QP and meeting AP in C . Then, letting $AP = r_1$ and $BQ = r_2$,

$$BC^2 = AB^2 - AC^2$$

$$= (r_1 + r_2)^2 - (r_1 - r_2)^2$$

$$= 4r_1 r_2$$

$$= 4b^2$$

$$\therefore BC = \pm 2b,$$

$$\therefore \tan \hat{ABC} = \pm \frac{2a}{2b} = \pm \frac{a}{b}$$

$$\therefore \tan 2\hat{ABC} = \pm \frac{2\frac{a}{b}}{1 - \frac{a^2}{b^2}} = \pm \frac{2ab}{b^2 - a^2}$$

$\therefore$ the tangent of the angle between the direct common tangents is $\pm \frac{2ab}{b^2 - a^2}$.

Example 3. A triangle ABC is right-angled at B and has given inradius r ; show that $AC = \frac{r}{\sqrt{2} \sin \frac{C}{2} \sin (45^\circ - \frac{C}{2})}$, and hence that

AC has minimum length when $C = 45^\circ$.

Let the incircle have centre I (Fig. 70) and touch AC at Y .

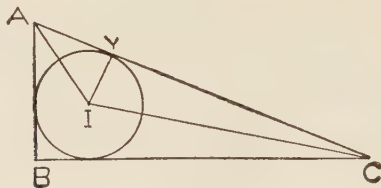


FIG. 70

Then $AC = AY + YC$

$$\begin{aligned}
 &= IY \cot \hat{IAY} + IY \cot \hat{ICY} \\
 &= r \cot \frac{A}{2} + r \cot \frac{C}{2} \\
 &= r \frac{\sin \frac{C}{2} \cos \frac{A}{2} + \cos \frac{C}{2} \sin \frac{A}{2}}{\sin \frac{C}{2} \sin \frac{A}{2}} \\
 &= r \frac{\sin \frac{C+A}{2}}{\sin \frac{C}{2} \sin \left(45^\circ - \frac{C}{2}\right)} \\
 &= \frac{r \sin 45^\circ}{\sin \frac{C}{2} \sin \left(45^\circ - \frac{C}{2}\right)} \\
 &= \frac{r}{\sqrt{2} \sin \frac{C}{2} \sin \left(45^\circ - \frac{C}{2}\right)}
 \end{aligned}$$

Hence $AC = \frac{\sqrt{2}r}{\cos(C - 45^\circ) - \cos 45^\circ}$, and this fraction has minimum value when its denominator has maximum value, i.e., when $C = 45^\circ$.

Example 4. From a coastguard station two ships at sea are observed to bear N.W. and N.E. respectively, while, from a second station one mile due east of the first, the ships are observed to bear W.N.W. and N.N.E. respectively. Find the distance between the ships.

Let **A** and **B** (Fig. 71) represent the stations, and **C** and **D** the ships ; let **AD** and **CD** be respectively x and y miles long.

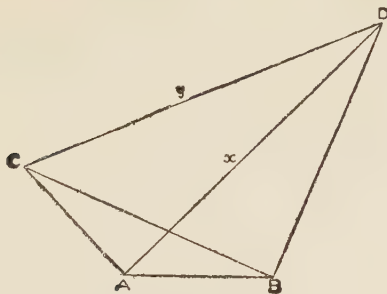


FIG. 71

$$\begin{aligned}\text{Then } \frac{AD}{AB} &= \frac{\sin \hat{A}BD}{\sin \hat{ADB}} \\ \therefore \frac{x}{1} &= \frac{\sin 112\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} \\ &= \frac{\cos 22\frac{1}{2}^\circ}{\sin 22\frac{1}{2}^\circ} \\ \therefore x &= \cot 22\frac{1}{2}^\circ\end{aligned}$$

But $\hat{CAD} = 90^\circ$ and $\hat{CBD} = 90^\circ$ and hence **A**, **B**, **D**, **C** are concyclic

$$\begin{aligned}\therefore \hat{CDA} &= \hat{CBA} \text{ and } \therefore \hat{CDA} = 22\frac{1}{2}^\circ \\ \therefore CD &= AD \sec 22\frac{1}{2}^\circ \\ \therefore y &= x \sec 22\frac{1}{2}^\circ \\ &= \cot 22\frac{1}{2}^\circ \sec 22\frac{1}{2}^\circ \\ &= \operatorname{cosec} 22\frac{1}{2}^\circ \\ &= 2.61\end{aligned}$$

$\therefore$ the distance between the ships is 2.61 miles.

Example 5. A tower of height h_1 is distance x due east of a second tower of height h_2 and standing on the same horizontal plane ; a point in this plane is S.W. of the first tower and S.E. of the other, and at this point the elevations of the tops of the towers are α and β respectively. Prove that

$$x = \sqrt{h_1^2 \cot^2 \alpha + h_2^2 \cot^2 \beta}$$

Let **A** and **B** (Fig. 72) represent the bases of the towers, and let **C** be the point at which the elevations are α and β .

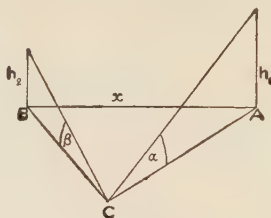


FIG. 72

Then $AC = h_1 \cot \alpha$ and $CB = h_2 \cot \beta$

But $\angle ACB = 90^\circ$

$$\therefore AB^2 = AC^2 + CB^2$$

$$\therefore x^2 = h_1^2 \cot^2 \alpha + h_2^2 \cot^2 \beta$$

$$\therefore x = \sqrt{h_1^2 \cot^2 \alpha + h_2^2 \cot^2 \beta}.$$

Example 6. A vertical rod 4 ft. long stands 7 ft. to the east of a wall running north and south. Two lamps, one east and the other south-east of the rod, cast shadows of the rod upon the wall; if these shadows are respectively 3 ft. and 1 ft. high, find the difference in the altitudes of the lamps as viewed from the top of the rod.

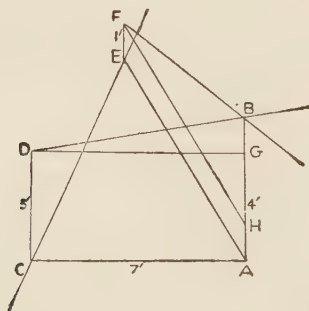


FIG. 73

Let **AB**, **CD**, **EF** (Fig. 73) represent the rod and the two shadows. Draw **DG** and **FH** perpendicular to **AB**.

$$\text{Then } \tan \hat{GDB} = \frac{GB}{DG} = \frac{AB - CD}{CA} = \frac{4 - 3}{7} = 0.1429$$

$$\therefore \hat{GDB} = 8^\circ 8'$$

$$\text{Also } \tan \hat{HFB} = \frac{HB}{FH} = \frac{AB - EF}{EA} = \frac{4 - 1}{7\sqrt{2}} = 0.3030$$

$$\therefore \hat{HFB} = 16^\circ 51'$$

$$\begin{aligned} \therefore \text{the difference in the altitudes of the lamps as viewed from B} \\ &= 16^\circ 51' - 8^\circ 8' \\ &= 8^\circ 43'. \end{aligned}$$

EXERCISES

1. From a ship at sea a lighthouse is observed to bear N.N.W., but after the ship has sailed 3 miles to the south-west the lighthouse is observed to be due north; find the distance of the lighthouse from the first point of observation.

2. From the top of a cliff a ship is observed to have angle of depression $8^\circ 22'$. If a second ship, one mile more distant, has angle of depression $3^\circ 10'$, find the height of the cliff above sea-level.

3. AB is a vertical line of length 100 feet; from a point C the angles of elevation of A and B are observed to be 24° and $22^\circ 30'$ respectively. Find the distance of C from the line AB .

4. From a point A two objects P and Q , lying on the horizontal plane through A , are observed to bear N.E.; from a point B in the same plane, and one mile east of A , P and Q bear N.N.W. and N.N.E. respectively. Find the distance between P and Q .

5. At A and B , two points on a vertical line, the angles of elevation of a point C are 37° and 48° respectively; at A , a point D vertically beneath C , has angle of depression 52° . Find the angle of elevation of B as viewed from D .

6. A and B are two points on a horizontal plane, and are 7 miles apart; A is west and B south of a mountain whose top is 4 and 5.8 miles from A and B respectively. Find the elevation of the mountain as viewed from A .

7. A vertical pole 80 feet high has angles of elevation 37° and 18° respectively when viewed from two points A and B lying in the horizontal plane through the foot of the pole. If B is east of A , and A is south of the pole, find the distance between A and B .

8. At two points A and B , in the same horizontal plane, a balloon

has angles of elevation 37° and 25° respectively; **A** is due south of the balloon, and **B** is east of **A**. Find the height of the balloon if the distance between **A** and **B** is 1000 feet.

9. The top of a vertical tower has angles of elevation 23° and 22° respectively at two points **A** and **B** lying in the horizontal plane through the base **C** of the tower. Find the height of the tower if $AB = 400$ feet and $\hat{BAC} = 90^\circ$.

10. Two vessels leave the same port at the same time; one steams N.W. at 14 knots, the other steams N.N.E. for 1 hour at 12 knots, then N.E. at 16 knots. Find the distance between the vessels 3 hours after leaving port.

11. **D** is vertically over and 8.32 feet above the vertex **A** of a horizontal triangle **ABC**; $\hat{DBA} = 20^\circ$, $\hat{DCA} = 27^\circ$, and $\hat{BDC} = 60^\circ$. Find the length of **BC**.

12. In a quadrilateral **ABCD**, $\hat{A} = \hat{D} = 90^\circ$, $AB = 3BC$, and $AD = 2DC$; show that $\hat{A} = 45^\circ$.

13. If a radius of a circle divides a chord in the ratio 2:1 and is bisected by the chord, show that the angle between the radius and the chord has cosine $\frac{1}{4}\sqrt{6}$.

14. A regular tetrahedron stands on a horizontal plane; prove that the slant edges are inclined to the horizontal at an angle whose cosine is $1/\sqrt{3}$.

15. In a horizontal triangle **APB**, $AP = 2AB$, and the angle **A** is a right angle. On **AB** stands a vertical and regular hexagon **ABCDEF**. Prove that **PC** is inclined to the horizontal at an angle whose tangent is $\sqrt{3}/5$.

16. At each of the vertices of a horizontal triangle **ABC**, a point **P**, 33 feet above the triangle, has angle of elevation $43^\circ 30'$; if $BC = 27$ feet, find the size of $\hat{BAC}$.

17. **PQ** is a horizontal line in a plane inclined at 32° to the horizontal; **PR** is a line in the plane and is inclined to **PQ** at 23° . Find the inclination of **PR** to the horizontal.

18. If θ is the angle in a segment of a circle of diameter d , show that the length of the chord which cuts off the segment is $d \sin \theta$.

19. In a horizontal triangle **ABC**, $BC = 2x$, $CA = 2y$, and $AB = 2z$; a vertical line **AD** subtends angle α at the mid-point of **BC**. Prove that

$$AD = \sqrt{2y^2 + 2z^2 - x^2} \tan \alpha.$$

20. If **A** is a point distant h from and vertically above a point **B**,

and if, at a third point **C**, **A** has angle of elevation α and **B** has angle of depression β , show that the difference of levels of **B** and **C** is

$$\frac{h \sin \beta \cos \alpha}{\sin (\alpha + \beta)}.$$

21. From a point **P**, two points **A** and **B** on a vertical line and distance h apart are observed to have angles of depression α and β respectively, α being greater than β . Show that the difference of levels of **P** and **A** is

$$\frac{h \sin \alpha \cos \beta}{\sin (\alpha - \beta)}.$$

22. **ABCD** is a horizontal rectangle; at a point vertically beneath **A** the angles of elevation of **B**, **C**, and **D** are α , β , and γ respectively. Prove that $\cot^2 \alpha + \cot^2 \gamma = \cot^2 \beta$.

23. **A**, **B**, **C**, **D** are coplanar points, and **AB** subtends angle α at **C** and **D**; show that $\sin \hat{\text{CAD}} = \frac{\text{CD}}{\text{AB}} \sin \alpha$.

24. Two intersecting circles, with radii a and b , have their centres distance c apart; prove that the cosine of the angle between the tangents at a point of intersection is $\pm \frac{a^2 + b^2 - c^2}{2ab}$.

25. **ABCD** is a horizontal rectangle; at a point distant h vertically beneath **A** the angles of elevation of **B** and **D** are α and β respectively. Show that the area of the rectangle is $h^2 \cot \alpha \cot \beta$.

26. In the triangle **ABC**, **BP** bisects $\hat{\text{B}}$, and **AP** is perpendicular to **BC**; prove that $\text{AP} = c \tan \frac{\text{B}}{2}$.

27. Two spheres of radii r_1 and r_2 touch each other and stand on a horizontal plane; show that the line of centres is inclined to the horizontal at angle θ , where $\cos \theta = \frac{2\sqrt{r_1 r_2}}{r_1 + r_2}$.

28. **AD** bisects $\hat{\text{A}}$ of triangle **ABC**, and meets **BC** in **D**; prove that

$$\text{AD} = \frac{2bc}{b+c} \cos \frac{\text{A}}{2}.$$

29. A vertical rod of length $2r \cos \theta$ stands at one end of a chord subtending angle 2θ at the centre of a horizontal circle of radius r ; show that the elevation of the top of the rod as viewed from the other end of the chord is $\frac{\pi}{2} - \theta$.

30. If two circles, with centres distance a apart, have the sum of

their radii equal to b , prove that the tangent of the angle between their transverse common tangents is $\pm \frac{2b\sqrt{a^2 - b^2}}{a^2 - 2b^2}$.

31. In a triangle ABC , AD , perpendicular to BC , meets BC in D ; if AD produced meets the circumcircle in E , show that

$$BD = c \cos B, \quad DC = b \cos C, \quad AD = b \sin C$$

and hence that $DE = 2R \cos B \cos C$.

32. $ABCD$ is a horizontal rectangle, whose diagonals intersect in O . At B , D , O , and C a vertical line AE subtends angles α , β , γ , and θ respectively; prove that $\cot^2 \alpha + \cot^2 \beta = \cot^2 \theta = 4 \cot^2 \gamma$.

33. ABO and CDO are two straight lines inclined to each other at angle 30° . $AB = BO = a$, and AB subtends angle 30° at C and D ; determine the length of CD .

34. From a point, distant d from the centre of a circle of radius r , tangents are drawn to the circle; if θ is the angle between the tangents show that $\tan \theta = \frac{2r\sqrt{d^2 - r^2}}{d^2 - 2r^2}$.

35. A is a point on the circumference of a circle, centre O and radius r . AB and AC are chords, one on either side of AO , and making with AO angles α and β respectively; prove that the length of BC is

$$2r\sqrt{\cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos (\alpha + \beta)}.$$

36. A , B , C are points in a horizontal plane; C is east and A north of B , while AB subtends angle α at C . Vertical towers of height x and y respectively are erected at A and B , the top of the tower at A having elevations β and γ respectively at C and at the top of the tower at B ; show that $\frac{y}{x} = 1 - \sin \alpha \cot \beta \tan \gamma$.

37. A vertical tower subtends angles α and β at two points A and B , distant x from each other, and lying in the horizontal plane through the base C of the tower. If $\hat{CAB} = \gamma$ and $\hat{CBA} = \delta$, show that the height of the tower is $x/\sqrt{\cot^2 \alpha + \cot^2 \beta + 2 \cot \alpha \cot \beta \cos (\gamma + \delta)}$.

38. A , B , C , D are consecutive vertices of a horizontal and regular hexagon. A vertical line DE subtends at A , B , and C angles α , β , and γ respectively; prove that $\cot^2 \alpha = \cot^2 \beta + \cot^2 \gamma$.

39. Two towers stand on a horizontal plane, the one being due south and half the height of the other. At a point due west of the higher tower the angle of elevation of the higher is α , of the lower β ; show that the elevation of the top of the higher tower as viewed from the top of the lower is γ , where $4 \cot^2 \alpha = \cot^2 \beta - \cot^2 \gamma$.

40. At two points **A** and **B** in the horizontal plane through the foot of a vertical pole the angles of elevation of the top of the pole are α and β respectively, while θ is the angle subtended at the foot of the pole by the line **AB**; show that the height of the pole is

$$AB/\sqrt{\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \theta}.$$

41. A vertical tower stands on a horizontal plane. Viewed from a point **A** due south, the elevation of the top of the tower is α , while from a point distance x east of **A** the elevation of the top is β . Prove that the height of the tower is $\frac{x \sin \alpha \sin \beta}{\sqrt{\sin(\alpha + \beta) \sin(\alpha - \beta)}}$.

42. The point **A** bears east and is distant x from **B**; a third point **C** is N.E. of **B** and y° east of north of **A**. Show that the distance between **B** and **C** is $\frac{\sqrt{2}x}{1 - \tan y^\circ}$.

43. From the top of a cliff a ship at sea is observed to have angle of depression α , while a balloon vertically above the ship has angle of elevation 2α ; if the height of the balloon above sea-level is x show that the height of the cliff is $\frac{1 - \tan^2 \alpha}{3 - \tan^2 \alpha} x$.

44. A vertical tower is built on a horizontal plane; to an observer standing on the plane the elevation of the top of the tower is α , but, when the observer has walked a distance x toward the tower, the elevation of the top of the tower is β . Neglecting the height of the observer, show that the height of the tower is $\frac{x \sin \alpha \sin \beta}{\sin(\beta - \alpha)}$.

45. **AB** is a given chord of a given circle, and **P** is a variable point on one of the arcs **AB**; show that the area of the triangle **ABP** varies as $\sin A \sin B$, and hence prove that when the area is a maximum the triangle is isosceles.

46. Two equal circles intersect at **B**, and **ABC** is a straight line meeting the circles at **A** and **C** respectively; show that the rectangle contained by **AB** and **BC** has maximum area when **AC** is parallel to the line of centres.

47. Two concentric circles have radii r_1 and r_2 respectively, r_1 being greater than r_2 ; a chord **AB** of the outer circle touches the inner circle. Show that the tangent of the angle subtended by **AB** at the common centre is $\frac{2r_2\sqrt{r_1^2 - r_2^2}}{2r_2^2 - r_1^2}$.

48. **ABCD** is a quadrilateral such that $\hat{A}B = a$, $\hat{B} = \hat{A}CD = 90^\circ$, and

$\hat{ACB} = \hat{CAD} = \theta$; prove that the area of the quadrilateral is

$$\frac{1}{2}a^2(\cot \theta + 2 \operatorname{cosec} 2\theta).$$

49. A straight line of length a lies in the horizontal plane through the base of a flag-pole of height a ; if the pole subtends angles α and β at the ends of the line, show that the line subtends at the base of the pole an angle whose cosine is

$$\frac{1}{2}(\tan \alpha \cot \beta + \cot \alpha \tan \beta - \tan \alpha \tan \beta).$$

50. The triangle ABC is right-angled at C ; O is a point on AC and distant x from C , while the circle with centre O and radius x touches AB . If $\tan \frac{A}{2} = t$, show that the length of AB is $\frac{x(1+t)(1+t^2)}{2t(1-t)}$.

51. If the radius of the circle inscribed in the triangle ABC has

$$\text{length } r, \text{ prove that } a = \frac{r \cos \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}.$$

52. The angle in the minor segment cut off from a circle by a chord AB , of length a , is ϕ . A tangent PQ meets BA produced in P and the perpendicular through B to the chord AB in Q . Show that, if $\hat{BPQ} = \theta$,

$$PQ = \frac{2a \cos^2 \frac{\phi - \theta}{2}}{\sin \phi \sin 2\theta}.$$

53. A ladder, inclined to the horizontal at angle ϕ , has its lower end fixed, while its upper end rests against a vertical wall. If the ladder were y units longer, the inclination to the horizontal would be θ ; show that the distance of the fixed end from the wall is $\frac{y \cos \theta \cos \phi}{\cos \phi - \cos \theta}$.

54. A chord of a circle of radius r has length $2l$, and touches a second circle drawn on a given radius of the first as diameter; show that the sine of the angle between the chord and the given radius is $1 - \frac{2}{r}\sqrt{r^2 - l^2}$.

55. AA' is a median of the triangle ABC , in which $BC = 4AB$; prove that $\tan \hat{AA'B} = \frac{\sin B}{2 - \cos B}$.

56. If the radii of two circles differ by a show that the intercepts made on the transverse common tangents by the direct common tangents have length $a \cot \frac{\theta}{2}$, where θ is the angle between the direct common tangents.

57. A and B are two points in a horizontal plane; B is due south

of and distant x from **A**. Two vertical towers on the same plane have heights y and z respectively, are due east of **A** and **B** respectively, and have elevations α and β at **A** and **B** respectively, while at the top of one tower the elevation of the top of the other is γ . Prove that

$$(y-z)^2 \cot^2 \gamma = x^2 + (y \cot \alpha - z \cot \beta)^2.$$

58. **A**, **B**, **C** are points in a vertical plane; **A** has elevations α and 2α as viewed from **B** and **C** respectively, while **C** has elevation β as viewed from **B**. If $BC = x$, show that the difference of levels of **A** and

B is either $x \sin (2\alpha - \beta)$ or $\frac{x \sin (2\alpha + \beta)}{2 \cos^2 \alpha + \cos 2\alpha}$.

59. **D** is a point on the side **BC** of a horizontal triangle **ABC**, and $BD = 2DC$. At **B**, **D**, and **C** the elevations of a point **P**, vertically over the point **A**, are α , β , and γ respectively. Show that

$$AP^2 : BC^2 = \frac{2}{3(\cot^2 \alpha - 3 \cot^2 \beta + 2 \cot^2 \gamma)}.$$

60. **ABCD** is a horizontal straight line, **AD** being trisected at **B** and **C**; at **A**, **B**, **C**, **D**, the elevations of a balloon are α , β , γ , δ respectively. Show that $\cot^2 \alpha - \cot^2 \delta = 3(\cot^2 \beta - \cot^2 \gamma)$.

61. A vertical tower subtends angles α and β respectively at two points **A** and **B**, which are in the horizontal plane through the base of the tower, and which are respectively south and west of the tower. Show that the cosine of the angle subtended at the top of the tower by the line **AB** is $\sin \alpha \sin \beta$.

62. A cube has one edge **AB** inclined at angle θ to the horizontal and another edge **AC** horizontal. If the diagonal through **A** is inclined to the horizontal at angle ϕ , prove that $\cos \phi = \sqrt{\frac{2}{3}} (1 - \sin \theta \cos \theta)$.

CHAPTER XIV

SOLUTION OF TRIGONOMETRICAL EQUATIONS

§ 52. If two expressions involving the circular functions of an angle are equal for certain values of that angle, the statement of this equality is called a *trigonometrical equation*. The determination of the values of the angle which satisfy an equation constitutes the solution of the equation.

The graphical solution of equations has been dealt with in Chapter VII; we shall now proceed to consider the analytical solution of equations of various types.

§ 53. *Type A.* Equations depending for their solution on the relations between the circular functions of an angle.

Example 1. Find the values of θ between 0° and 360° , which satisfy the following equations :

$$(a) \ 2 \sin \theta = \sqrt{3}, \quad (b) \ 4 \cos \theta = 3 \sec \theta.$$

$$(a) \ 2 \sin \theta = \sqrt{3}$$

$$\therefore \sin \theta = \frac{\sqrt{3}}{2}$$

$$\therefore \theta = 60^\circ \text{ or } 120^\circ.$$

$$(b) \ 4 \cos \theta = 3 \sec \theta$$

$$\therefore 4 \cos \theta = \frac{3}{\cos \theta}$$

$$\therefore 4 \cos^2 \theta = 3$$

$$\therefore \cos^2 \theta = \frac{3}{4}$$

$$\therefore \cos \theta = \pm \frac{\sqrt{3}}{2}$$

$$\therefore (i) \cos \theta = \frac{\sqrt{3}}{2}$$

$$\text{or } (ii) \cos \theta = -\frac{\sqrt{3}}{2}$$

$$\therefore \theta = 30^\circ \text{ or } 360^\circ - 30^\circ$$

$$\therefore \theta = 180^\circ - 30^\circ \text{ or } 180^\circ + 30^\circ$$

$$= 30^\circ \text{ or } 330^\circ$$

$$= 150^\circ \text{ or } 210^\circ$$

$$\therefore \text{the values of } \theta \text{ are } 30^\circ, 150^\circ, 210^\circ, 330^\circ.$$

Example 2. For the same range of values solve the equation

$$6 \sec^2 x - \tan x - 8 = 0.$$

$$6 \sec^2 x - \tan x - 8 = 0$$

$$\therefore 6 (\tan^2 x + 1) - \tan x - 8 = 0$$

$$\therefore 6 \tan^2 x - \tan x - 2 = 0$$

$$\therefore (3 \tan x - 2)(2 \tan x + 1) = 0$$

$$\therefore \text{(i) } 3 \tan x - 2 = 0$$

$$\therefore \tan x = \frac{2}{3}$$

$$= 0.6667$$

$$\therefore x = 33^\circ 42' \text{ or } 180^\circ + 33^\circ 42'$$

$$= 33^\circ 42' \text{ or } 213^\circ 42'$$

$$\text{or (ii) } 2 \tan x + 1 = 0$$

$$\therefore \tan x = -\frac{1}{2}$$

$$= -0.5000$$

$$\therefore x = 180^\circ - 26^\circ 34' \text{ or } 360^\circ - 26^\circ 34'$$

$$= 153^\circ 26' \text{ or } 333^\circ 26'$$

$\therefore$ the values of x are $33^\circ 42'$, $153^\circ 26'$, $213^\circ 42'$, $333^\circ 26'$.

Example 3. Solve, giving the values of θ between 0° and 180° , the equation

$$\cot \theta + 4 \tan \theta = 4 \operatorname{cosec} \theta.$$

$$\cot \theta + 4 \tan \theta = 4 \operatorname{cosec} \theta$$

$$\therefore \frac{\cos \theta}{\sin \theta} + 4 \frac{\sin \theta}{\cos \theta} = \frac{4}{\sin \theta}$$

Multiplying both sides by $\cos \theta \sin \theta$, we have

$$\cos^2 \theta + 4 \sin^2 \theta = 4 \cos \theta$$

$$\therefore \cos^2 \theta + 4(1 - \cos^2 \theta) = 4 \cos \theta$$

$$\therefore 3 \cos^2 \theta + 4 \cos \theta - 4 = 0$$

$$\therefore (3 \cos \theta - 2)(\cos \theta + 2) = 0$$

$$\therefore \text{(i) } 3 \cos \theta - 2 = 0$$

$$\therefore \cos \theta = \frac{2}{3}$$

$$= 0.6667$$

$$\therefore \theta = 48^\circ 11'$$

$$\text{or (ii) } \cos \theta + 2 = 0$$

$$\therefore \cos \theta = -2$$

This equation is inadmissible, since $\cos \theta$ cannot be numerically greater than 1.

$\therefore$ the value of θ is $48^\circ 11'$.

Note. These examples illustrate the common methods used to solve equations of this type. We employ any of the relations between the circular functions of an angle in order to reduce the

given equation to one in which there appears only one circular function. Note that if in the given equation there is only one term of the first degree the solution is generally obtained by expressing the other functions in terms of the one appearing in the first degree.

Example 4. Give the general solution of the following equations :

$$(a) \sin \theta = \frac{1}{\sqrt{2}}, \quad (b) \cos \theta = \frac{1}{2}, \quad (c) \tan \theta = \frac{1}{\sqrt{3}}.$$

$$(a) \sin \theta = \frac{1}{\sqrt{2}}$$

$\therefore \theta = 45^\circ$ or $180^\circ - 45^\circ$ (if θ is positive and not greater than 360°)

Let $\hat{XOP} = 45^\circ$, $\hat{XOP}_1 = 180^\circ - 45^\circ$ (Fig. 74). Then OX , OP or OX , OP_1 are the arms of *all* angles satisfying the equation.

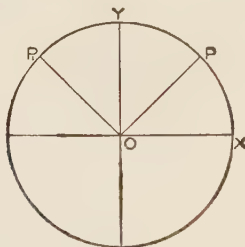


FIG. 74

But OX , OP are the arms of the angles $2n \cdot 180^\circ + 45^\circ$
and OX , OP_1 „ „ „ $(2n+1) 180^\circ - 45^\circ$
where n is an integer, positive or negative.

$$\therefore \theta = 2n \cdot 180^\circ + 45^\circ \quad \text{or} \quad (2n+1) 180^\circ - 45^\circ.$$

If the angles are expressed in circular measure,

$$\theta = 2n\pi + \frac{\pi}{4} \quad \text{or} \quad (2n+1)\pi - \frac{\pi}{4}.$$

Note 1. n being integral, $2n \cdot 180^\circ$ is an even multiple of 180° and $(2n+1) 180^\circ$ is an odd multiple of 180° .

Note 2. By substituting particular values for n in the general

solution, we obtain particular values of θ which satisfy the equation

$$\sin \theta = \frac{1}{\sqrt{2}};$$

$$\text{e.g., if } n=0, \quad \theta = 45^\circ \text{ or } 135^\circ$$

$$\text{and if } n=-1, \theta = -315^\circ \text{ or } -225^\circ.$$

Note 3. General solutions may be expressed in different forms :

e.g., $\theta = \frac{n\pi}{2}$ and $\theta = n\pi \pm \frac{\pi}{4}$ together give the same series of values

for θ as does $\theta = \frac{n\pi}{4}$.

$$(b) \cos \theta = \frac{1}{2}$$

$$\therefore \theta = 60^\circ \text{ or } 360^\circ - 60^\circ \text{ (if } \theta \text{ is positive and not greater than } 360^\circ)$$

Let $\hat{XOP} = 60^\circ$, $\hat{XOP}_1 = 360^\circ - 60^\circ$ (Fig. 75). Then OX , OP or OX , OP_1 are the arms of *all* angles satisfying the equation.

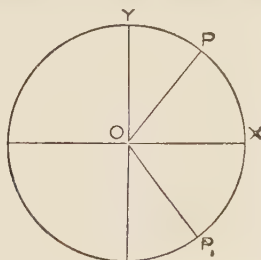


FIG. 75

But OX , OP are the arms of the angles $n \cdot 360^\circ + 60^\circ$
and OX , OP_1 „ „ „ „ $n \cdot 360^\circ - 60^\circ$
where n is an integer, positive or negative.

$$\therefore \theta = n \cdot 360^\circ \pm 60^\circ.$$

If the angles are expressed in circular measure,

$$\theta = 2n\pi \pm \frac{\pi}{3}.$$

$$(c) \tan \theta = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = 30^\circ \text{ or } 180^\circ + 30^\circ \text{ (if } \theta \text{ is positive and not greater than } 360^\circ)$$

14. $2 \sin^2 \theta + \sin \theta = 1$.
15. $12 \cos^2 x + \cos x - 6 = 0$.
16. $14 \tan^2 x = 11 \tan x + 15$.
17. $\sqrt{3} (\tan^2 \theta + 1) = 4 \tan \theta$.
18. $2 \sec^2 x - 13 \sec x + 20 = 0$.
19. $5 \cos^2 \theta - 2 \cos \theta = 3 \sin^2 \theta$.
20. $5 \sec^2 \theta + 9 \tan \theta = 1$.
21. $6 \cos^2 x - 7 \sin x = 8$.
22. $10 \sin^2 x + 3 \cos x = 6$.
23. $3 \sec^2 \theta - 5 \tan \theta = 5$.
24. $6 \sin x \cos x + 3 \sin x = 2 \cos x + 1$.
25. $8 \cos^4 \theta - 10 \cos^2 \theta + 3 = 0$.
26. $3 \tan \theta - \cot \theta = 5$.
27. $3 \sin^2 x + 7 \sin x - 1 = 0$.
28. $2 \cos^2 x - 5 \cos x = 5$.
29. $5 \sin^2 \theta - 4 \sin \theta \cos \theta + 3 \cos^2 \theta = 2$.
30. $6 \sin^2 \theta + 5 \sin \theta \cos \theta - 5 \cos^2 \theta = 1$.
31. $3 \tan \theta - 2 \cot \theta + 3 \sec \theta = 0$.
32. $5 \cot \theta + 2 \tan \theta = 5 \operatorname{cosec} \theta$.
33. $\cot \theta + 3 \tan \theta = 5 \operatorname{cosec} \theta$.
34. $6 \tan \theta - \sec \theta = 6 \cot \theta$.
35. $2 \sin x + 2\sqrt{3} \cos x = \sqrt{3} \tan x + 3$.
36. $\sqrt{6} \cos x + \sqrt{2} \sin x + \sqrt{3} \cot x + 1 = 0$.
37. $20 \cot x + 15 \cot x \operatorname{cosec} x - 4 \operatorname{cosec} x = 3 (1 + \cot^2 x)$.
38. $\sqrt{2} + 2\sqrt{3} \sin x + \sqrt{6} \cot x = 2\sqrt{3} \operatorname{cosec} x + 2 \cos x$.

Give the general solution of the following equations :

39. $2 \sin \theta = -1$.
40. $\cos^2 \theta = \frac{3}{4}$.
41. $2 \cos^2 \theta = \cos \theta$.
42. $2 \sin \theta \cos \theta = \sqrt{3} \sin \theta$.
43. $\operatorname{cosec}^2 \theta = 2$.
44. $4 \sin \theta \cos \theta + 1 = 2 (\cos \theta + \sin \theta)$.
45. $\tan \theta + \cot \theta = 2$.
46. $3 \sin \theta = 4 \sin^3 \theta$.
47. $4 \cos^3 \theta = 3 \cos \theta$.
48. $\sec^2 \theta = 3 \tan \theta - 1$.
49. $8 \cos^2 \theta + 10 \sin \theta = 11$.
50. $\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} + \cos \theta = 0$.
51. $(\sqrt{3} + 1) \cos^2 \theta - 1 = (\sqrt{3} - 1) \sin \theta \cos \theta$.
52. $\frac{1 + 2 \sin^2 \theta}{\cos^2 \theta} + 4 \tan \theta = 0$.

§ 54. *Type B. Equations involving multiple-angle and product formulæ.*

Example 1. Solve the following equations, giving the values of θ from 0° to 180° , inclusive :

$$(a) \cos 2\theta = \frac{1}{2}, \quad (b) \sin 2\theta + \sin 4\theta = \sin 3\theta.$$

$$(a) \cos 2\theta = \frac{1}{2}$$

$$\therefore 2\theta = 60^\circ \text{ or } 300^\circ$$

$$\therefore \theta = 30^\circ \text{ or } 150^\circ.$$

$$(b) \sin 2\theta + \sin 4\theta = \sin 3\theta$$

$$\therefore 2 \sin 3\theta \cos \theta - \sin 3\theta = 0$$

$$\therefore \sin 3\theta (2 \cos \theta - 1) = 0$$

$$\therefore (i) \sin 3\theta = 0$$

$$\text{or } (ii) 2 \cos \theta - 1 = 0$$

$$\therefore 3\theta = 0^\circ, 180^\circ, 360^\circ, 540^\circ$$

$$\therefore \cos \theta = \frac{1}{2}$$

$$\therefore \theta = 0^\circ, 60^\circ, 120^\circ, 180^\circ$$

$$\therefore \theta = 60^\circ$$

$$\therefore \text{the values of } \theta \text{ are } 0^\circ, 60^\circ, 120^\circ, 180^\circ.$$

Note 1. θ is restricted to values not greater than 180° ; hence for 2θ we determine values not greater than twice 180° and for 3θ values not greater than three times 180° .

Note 2. In Example 1 (b) it is a common error to express $\sin 2\theta + \sin 4\theta$ as a product, divide the equation by $\sin 3\theta$, and ignore the equation $\sin 3\theta = 0$. If an equation is divided by an expression involving the variable (*e.g.*, $\sin 3\theta$) then that expression equated to zero may give one or more roots of the equation.

Example 2. Solve the equation $\cos 3\theta + \sin 3\theta = \cos \theta + \sin \theta$.

$$\cos 3\theta + \sin 3\theta = \cos \theta + \sin \theta$$

$$\therefore \sin 3\theta - \sin \theta = \cos \theta - \cos 3\theta$$

$$\therefore 2 \cos 2\theta \sin \theta = 2 \sin 2\theta \sin \theta$$

$$\therefore 2 \sin \theta (\cos 2\theta - \sin 2\theta) = 0$$

$$\therefore (i) \sin \theta = 0$$

$$\text{or } (ii) \cos 2\theta - \sin 2\theta = 0$$

$$\therefore \theta = n\pi$$

$$\therefore \cos 2\theta = \cos \left(\frac{\pi}{2} - 2\theta \right)$$

$$\therefore 2\theta = 2n\pi \pm \left(\frac{\pi}{2} - 2\theta \right)$$

$$\therefore 4\theta = 2n\pi + \frac{\pi}{2}$$

$$\therefore \theta = \frac{n\pi}{2} + \frac{\pi}{8}$$

$\therefore$ the general solution is

$$\theta = n\pi$$

$$\text{or } \theta = n\frac{\pi}{2} + \frac{\pi}{8}.$$

Note. The following gives an alternative solution of the equation $\cos 2\theta - \sin 2\theta = 0$.

$$\cos 2\theta - \sin 2\theta = 0$$

$$\therefore \tan 2\theta = 1$$

$$\therefore 2\theta = n\pi + \frac{\pi}{4}$$

$$\therefore \theta = n\frac{\pi}{2} + \frac{\pi}{8}.$$

EXERCISES

Find the values of the angles between 0° and 360° , which satisfy the following equations :

1. $\cos 2\theta = -\frac{\sqrt{3}}{2}.$
2. $\tan 2\theta = -\frac{1}{\sqrt{3}}.$
3. $\sin \frac{\theta}{2} = \frac{1}{2}.$
4. $\sin \frac{3\theta}{2} = -\frac{1}{\sqrt{2}}.$
5. $3 \sin 2x = 2 \sin x.$
6. $\tan 2\theta + \tan \theta = 0.$
7. $\sin 3\theta + \sin \theta = \sqrt{3} \sin 2\theta.$
8. $\cos \theta + \cos 2\theta + \cos 3\theta = 0.$
9. $\sin 3\theta + \cos 3\theta = \cos \theta - \sin \theta.$
10. $6 \cos \theta + 5 \sin \frac{\theta}{2} = 3.$
11. $\sin \frac{\theta}{3} - 3 \sin \theta = 0.$
12. $\sin 3\theta - \sin \theta = \cos 4\theta + 1.$
13. $\sin 4x + 2 \sin 2x = 2 \cos^2 x.$
14. $5 \cos 2\theta + 2 \cos \frac{2\theta}{3} = 0.$
15. $3 \cos 2\theta = 2 + \cos \theta.$
16. $7 \sin^2 2\theta - 4 \sin 4\theta + 5 \cos^2 2\theta = 2.$
17. $\sin 3\theta + \cos^2 \theta = \sin \theta + 2 \cos^4 \theta.$
18. $\cos \theta (3 \sin \theta - 1) = 4 \cos \theta \sin^3 \theta - \cos 5\theta.$

Find the values of the angles from 0° to 180° , inclusive, which satisfy the following equations :

19. $\sin 3\theta = \frac{\sqrt{3}}{2}.$
20. $\cos 4\theta = 1.$
21. $3 \cos 2\theta = 2 \sin \theta \cos \theta.$
22. $\cos 2\theta - \cos 4\theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}.$
23. $\cos \theta = \cos 2\theta + \sin 3\theta.$
24. $\sin 7x + \sin 3x = \sqrt{2} \cos 2x.$
25. $\sin \theta - \sin 3\theta + \sin 5\theta = 0.$
26. $\cos 2\theta \cos \theta = \sin 4\theta \sin \theta.$

$$27. 2 \sin \theta \left(\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} \right) = \sin 5\theta - \sin \theta.$$

$$28. 10 \sin^2 2\theta - \sin 4\theta = 2 - \cos^2 2\theta.$$

$$29. \sin 5\theta - \sin 3\theta = \cos 6\theta + \cos 2\theta.$$

$$30. \cos^2 2\theta - \sin^2 \theta = \sin 4\theta - \sin 2\theta.$$

Give the general solution of the following equations :

$$31. \cos 2\theta = \cos \alpha.$$

$$32. \cos 3\theta = -\cos \beta.$$

$$33. \sin 3\theta = \cos 2\theta.$$

$$34. \sin 5\theta + \sin \theta = 2 \cos 4\theta \sin \theta.$$

$$35. 5 \cos 2\theta = 2 - \sin \theta.$$

$$36. \sin 3\theta - \sqrt{3} \cos \theta = \sin \theta + \sqrt{3} \cos 3\theta.$$

$$37. \cos \theta + \cos 2\theta + \cos 3\theta + 1 = 0.$$

$$38. \tan 3\theta - \tan 2\theta - \tan \theta = 0.$$

$$39. \sin (m+1)\theta - \sin (m-1)\theta = \sqrt{3} \sin \theta.$$

$$40. \cos (p-q)\theta - \cos (p+q)\theta = \sin 2q\theta.$$

§ 55. *Type C.* Equations of the form $a \cos \theta \pm b \sin \theta = \pm c$ where a , b , and c are positive quantities.

These equations can be solved if c is not greater than $\sqrt{a^2 + b^2}$, and the methods of solution are indicated in the following examples.

Example 1. Find the values of θ between 0° and 360° which satisfy the equation

$$12 \cos \theta + 5 \sin \theta = 3.$$

$$12 \cos \theta + 5 \sin \theta = 3$$

Dividing each term of the equation by $\sqrt{12^2 + 5^2}$, we have

$$\frac{12}{13} \cos \theta + \frac{5}{13} \sin \theta = \frac{3}{13}$$

Let α be the acute angle such that

$$\cos \alpha = \frac{12}{13} \text{ and } \sin \alpha = \frac{5}{13}$$

$$\therefore \cos \alpha \cos \theta + \sin \alpha \sin \theta = \frac{3}{13}$$

$$\therefore \cos (\theta - \alpha) = \frac{3}{13} = 0.2308$$

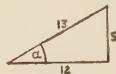
$$\therefore \theta - \alpha = 76^\circ 39' \text{ or } 283^\circ 21'$$

$$\therefore \theta = 76^\circ 39' + 22^\circ 37'$$

$$\text{or } 283^\circ 21' + 22^\circ 37'$$

$$\therefore \theta = 99^\circ 16' \text{ or } 305^\circ 58'$$

$$\therefore \text{the values of } \theta \text{ are } 99^\circ 16', 305^\circ 58'.$$



$$\begin{aligned} \tan \alpha &= \frac{5}{12} \\ &= 0.4167 \\ \therefore \alpha &= 22^\circ 37' \end{aligned}$$

Example 2. For the same range of values, solve the equation

$$4 \sin \theta - 5 \cos \theta = 2.$$

$$4 \sin \theta - 5 \cos \theta = 2$$

$$\therefore 5 \cos \theta - 4 \sin \theta = -2$$

Dividing each term of the equation by $\sqrt{5^2 + 4^2}$, we have

$$\frac{5}{\sqrt{41}} \cos \theta - \frac{4}{\sqrt{41}} \sin \theta = -\frac{2}{\sqrt{41}}$$

Let α be the acute angle such that

$$\cos \alpha = \frac{5}{\sqrt{41}} \text{ and } \sin \alpha = \frac{4}{\sqrt{41}}$$

$$\therefore \cos \alpha \cos \theta - \sin \alpha \sin \theta = -\frac{2}{\sqrt{41}}$$

$$\therefore \cos (\theta + \alpha) = -\frac{2}{\sqrt{41}}$$

$$\therefore \theta + \alpha = 180^\circ - 71^\circ 48'$$

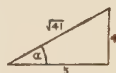
$$\text{or } 180^\circ + 71^\circ 48'$$

$$\therefore \theta = 108^\circ 12' - 38^\circ 40'$$

$$\text{or } 251^\circ 48' - 38^\circ 40'$$

$$\therefore \theta = 69^\circ 32' \text{ or } 213^\circ 8'$$

$$\therefore \text{the values of } \theta \text{ are } 69^\circ 32', 213^\circ 8'.$$



$$\tan \alpha = \frac{4}{5}$$

$$= 0.8000$$

$$\therefore \alpha = 38^\circ 40'$$

Nos.	Logs.
2	0.3010
$\sqrt{41}$	0.8064
$\log\{-\cos(\theta + \alpha)\}$	1.4946

Note 1. In Example 2, the equation is rewritten so that the term involving $\cos \theta$ has a positive sign. The equation is thus reduced to one involving $\cos (\theta \pm \alpha)$, where α is acute.

Note 2. Since $\cos (\theta \pm \alpha)$ cannot be greater than one, the student will now see that equations of the form $a \cos \theta \pm b \sin \theta = \pm c$, cannot be solved if c is greater than $\sqrt{a^2 + b^2}$.

Example 3. Solve the equation $\sin 2\theta - \sqrt{3} \cos 2\theta = 1$.

$$\sin 2\theta - \sqrt{3} \cos 2\theta = 1$$

$$\therefore \frac{\sqrt{3}}{2} \cos 2\theta - \frac{1}{2} \sin 2\theta = -\frac{1}{2}$$

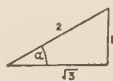
Let α be the acute angle whose cosine is $\frac{\sqrt{3}}{2}$ and whose sine is $\frac{1}{2}$

$$\therefore \cos \alpha \cos 2\theta - \sin \alpha \sin 2\theta = -\frac{1}{2}$$

$$\therefore \cos (2\theta + \alpha) = -\frac{1}{2}$$

$$\therefore 2\theta + \alpha = 2n\pi \pm \frac{2\pi}{3}$$

$$\therefore 2\theta = 2n\pi \pm \frac{2\pi}{3} - \frac{\pi}{6}$$



$$\tan \alpha = \frac{1}{\sqrt{3}}$$

$$\therefore \alpha = \frac{\pi}{6}$$

$$\begin{aligned}
 \therefore \frac{6}{y} - 2 + \frac{2}{y^2} &= 2 + \frac{2}{y^2} \\
 \therefore 6y - 2y^2 + 2 &= 2y^2 + 2 \\
 \therefore 4y^2 - 6y &= 0 \\
 \therefore 2y(2y - 3) &= 0 \\
 \therefore (i) y = 0 &\quad \text{or (ii) } 2y - 3 = 0 \\
 \therefore t = \infty &\quad \therefore y = \frac{3}{2} \\
 \therefore \frac{\theta}{2} = 90^\circ &\quad \therefore t = \frac{2}{3} \\
 &\quad = 0.6667 \\
 \therefore \theta = 180^\circ &\quad \therefore \frac{\theta}{2} = 33^\circ 42' \\
 &\quad \therefore \theta = 67^\circ 24' \\
 \therefore \text{the values of } \theta &\text{ are } 67^\circ 24', 180^\circ.
 \end{aligned}$$

Note 1. The quadratic equation (i) may be rewritten

$$0 \cdot t^2 + 3t - 2 = 0,$$

and has one root infinite; the substitution $t = \frac{1}{y}$ is unnecessary if we remember this infinite root.

Note 2. In Example 6, the equation of Example 5 is solved by another method.

Example 6. Solve the following equation, giving the values of θ between 0° and 360° :

$$3 \sin \theta - 2 \cos \theta = 2.$$

$$3 \sin \theta - 2 \cos \theta = 2$$

Squaring both sides of the equation, we have

$$9 \sin^2 \theta + 4 \cos^2 \theta - 12 \sin \theta \cos \theta = 4 (\cos^2 \theta + \sin^2 \theta)$$

$$\therefore 5 \sin^2 \theta - 12 \sin \theta \cos \theta = 0$$

$$\therefore \sin \theta (5 \sin \theta - 12 \cos \theta) = 0$$

$$\therefore (i) \sin \theta = 0 \quad \text{or (ii) } 5 \sin \theta = 12 \cos \theta$$

$$\therefore \theta = 180^\circ \quad \therefore \tan \theta = \frac{12}{5}$$

$$= 2.4000$$

$$\therefore \theta = 67^\circ 23' \text{ or } 247^\circ 23'$$

On substitution, it is found that $67^\circ 23'$ and 180° satisfy the given equation, and that $247^\circ 23'$ does not.

$$\therefore \text{the values of } \theta \text{ are } 67^\circ 23', 180^\circ.$$

Note 1. If this method of solution is used it is necessary to check the results by substitution in the given equation. It may

be noted that the values 0° and $247^\circ 23'$ satisfy the equation $3 \sin \theta - 2 \cos \theta = -2$.

Note 2. When both sides of the equation are squared and the equation is made homogeneous by multiplying the constant term by $\cos^2 \theta + \sin^2 \theta$, the resultant equation generally reduces to a quadratic equation in $\tan \theta$.

EXERCISES

Find the values of the angles between 0° and 360° , which satisfy the following equations (use the first method of the worked examples):

- | | |
|--|--|
| 1. $3 \cos \theta + 4 \sin \theta = 5$. | 6. $9 \cos \theta + 7 \sin \theta = 3$. |
| 2. $5 \cos \theta + 12 \sin \theta = 13$. | 7. $6 \sin \theta = 5 \cos \theta + 7$. |
| 3. $4 \sin \theta - 3 \cos \theta = 2$. | 8. $\sqrt{2} = \cos 3\theta + \sin 3\theta$. |
| 4. $12 \sin \theta - 5 \cos \theta = 9$. | 9. $3 \cos 3\theta = \sqrt{13} - 2 \sin 3\theta$. |
| 5. $7 \cos \theta + 4 \sin \theta = 5$. | 10. $4 \sin 2\theta - 3 \cos 2\theta = 1$. |

For the same range of values, solve the following equations by using the substitutions $\sin \theta = \frac{2t}{1+t^2}$, $\cos \theta = \frac{1-t^2}{1+t^2}$, where $t = \tan \frac{\theta}{2}$ (check Nos. 19 and 20 by an alternative method):

- | | |
|---|---|
| 11. $7 \cos \theta - 6 \sin \theta = 2$. | 16. $\sin 3x - 10 = 18 \cos 3x$. |
| 12. $4 \cos \theta + 5 \sin \theta = 5$. | 17. $8 \sin \theta - 7 \cos \theta = 2$. |
| 13. $3 \sin \theta = 3 + 5 \cos \theta$. | 18. $15 \cos \theta - 7 = 10 \sin \theta$. |
| 14. $9 \cos \theta = 6 - 2 \sin \theta$. | 19. $4(1 + \cos \theta) = 5 \sin \theta$. |
| 15. $8 = 13 \cos 2x + 16 \sin 2x$. | 20. $7 \sin \theta = 5(1 + \cos \theta)$. |

Give the general solution of the following equations:

21. $\sqrt{3} \cos \theta - \sin \theta = 1$.
22. $\sqrt{3} \sin 2\theta = 2 + \cos 2\theta$.
23. $\sin 4\theta + \cos 4\theta - 1 = 0$.
24. $\sqrt{2} (\cos \theta + \sin \theta) = \sqrt{3}$.
25. $2 \sin 3\theta - \sqrt{2} = 2 \cos 3\theta$.
26. $5 \cos \frac{\theta}{2} + 5 \sin \frac{\theta}{2} = \sqrt{2}$.
27. $\sqrt{3} (\cos m\theta - 1) = \sin m\theta$.
28. $(\sqrt{3} + 1) \cos 2\theta + (\sqrt{3} - 1) \sin 2\theta = 2$.
29. $(\sqrt{3} + 1) \sin \theta - 2\sqrt{2} = (\sqrt{3} - 1) \cos \theta$.
30. $\cos (\theta + \beta) - \sqrt{2} \cos \alpha = -\sin (\theta + \beta)$.

CHAPTER XV

PROPERTIES OF TRIANGLES

§ 56. In Chapter X we proved a number of identities concerning the sides and angles of triangles ; in this chapter we shall deal with some further properties of triangles. Before proceeding with this chapter the student should be familiar with the following geometrical theorems :

I. The right bisectors of the three sides of a triangle meet in a point which is the centre of the circle passing through the three vertices of the triangle.

The circle, its centre, and its radius are called respectively the circumcircle, circumcentre, and circumradius of the triangle.

II. The internal bisectors of the three angles of a triangle meet in a point which is the centre of the circle which touches the three sides of the triangle and lies within the triangle.

The circle, its centre, and its radius are called respectively the incircle, incentre, and inradius of the triangle.

III. The internal bisector of an angle of a triangle and the external bisectors of the other two angles meet in a point which is the centre of a circle touching one side of the triangle and the other two sides produced.

Such a circle, its centre, and its radius are called respectively an excircle, an excentre, and an exradius of the triangle.

IV. The three lines drawn through the vertices of a triangle and perpendicular to the opposite sides are concurrent.

The three lines and their point of concurrence are called respectively the perpendiculars and the orthocentre of the triangle, while the triangle whose vertices are the feet of the perpendiculars is called the pedal triangle of the original triangle.

§ 57. NOTATION

We shall adopt the notation of Chapter X ; also we shall denote

the circumcentre by O ,

the incentre and inradius by I and r ,

the points at which the incircle touches the sides a, b, c by X, Y, Z ,

the centres and radii of the excircles opposite the angles A, B, C

by I_1, I_2, I_3 and r_1, r_2, r_3 ,

the points at which these circles touch the side a by X_1, X_2, X_3 ,

the side b by Y_1, Y_2, Y_3 , and the side c by Z_1, Z_2, Z_3 ,

the feet of the perpendiculars to the sides a, b, c by D, E, F ,

and the orthocentre by H .

§ 58. To prove that in any triangle $R = \frac{abc}{4\Delta}$.

$$\begin{aligned} R &= \frac{a}{2 \sin A} \\ &= \frac{abc}{4 \cdot \frac{1}{2} bc \sin A} \\ &= \frac{abc}{4\Delta}. \end{aligned}$$

§ 59. To prove that in any triangle

$$(i) \ r = \frac{\Delta}{s}$$

$$(ii) \ r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$$

$$(iii) \ r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

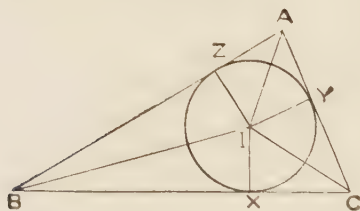


FIG. 77

In Fig. 77, (i) $\Delta ABC = \Delta BIC + \Delta CIA + \Delta AIB$

$$= \frac{1}{2} BC \cdot IX + \frac{1}{2} CA \cdot IY + \frac{1}{2} AB \cdot IZ$$

$$= \frac{1}{2} ar + \frac{1}{2} br + \frac{1}{2} cr$$

$$= r \frac{a+b+c}{2}$$

$$= rs$$

$$\therefore r = \frac{\Delta}{s}.$$

$$\begin{aligned}
 \text{(ii) } 2s &= AY + AZ + BZ + BX + CX + CY \\
 &= 2AZ + 2BX + 2CX \\
 &= 2AZ + 2BC \\
 &= 2AZ + 2a
 \end{aligned}$$

$$\therefore AZ = (s - a)$$

$$\text{But } IZ = AZ \tan \hat{ZAI}$$

$$\therefore r = (s - a) \tan \frac{A}{2},$$

$$\text{and similarly } r = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2}.$$

$$\text{(iii) } a = BC$$

$$= BX + XC$$

$$= IX \cot \hat{XBI} + IX \cot \hat{XCI}$$

$$= r \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right)$$

$$= r \frac{\cos \frac{B}{2} \sin \frac{C}{2} + \cos \frac{C}{2} \sin \frac{B}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}$$

$$= r \frac{\sin \frac{B+C}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}$$

$$= r \frac{\cos \frac{A}{2}}{\sin \frac{B}{2} \sin \frac{C}{2}}$$

$$\therefore r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$$

$$= \frac{2R \sin A \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$$

$$= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

§ 60. To prove that in any triangle

$$\text{(i) } r_1 = \frac{\Delta}{s - a}$$

$$\begin{aligned}
 \text{(ii) } r_1 &= s \tan \frac{A}{2} \\
 &= (s - c) \cot \frac{B}{2} = (s - b) \cot \frac{C}{2} \\
 \text{(iii) } r_1 &= 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.
 \end{aligned}$$

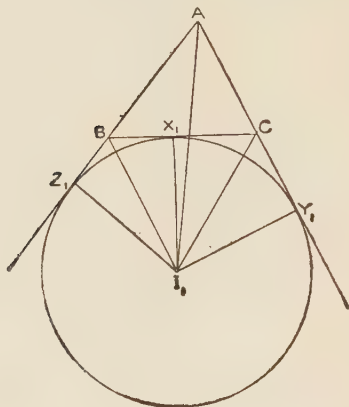


FIG. 78

In Fig. 78, (i) $\Delta ABC = \Delta CI_1A + \Delta AI_1B - \Delta BI_1C$

$$\begin{aligned}
 &= \frac{1}{2} CA \cdot I_1Y_1 + \frac{1}{2} AB \cdot I_1Z_1 - \frac{1}{2} BC \cdot I_1X_1 \\
 &= \frac{1}{2} br_1 + \frac{1}{2} cr_1 - \frac{1}{2} ar_1 \\
 &= r_1 \frac{b + c - a}{2} \\
 &= r_1 (s - a) \\
 \therefore r_1 &= \frac{\Delta}{s - a}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } 2s &= AB + BX_1 + AC + CX_1 \\
 &= AB + BZ_1 + AC + CY_1 \\
 &= AZ_1 + AY_1 \\
 &= 2AZ_1 \\
 \therefore AZ_1 &= s
 \end{aligned}$$

But $I_1Z_1 = AZ_1 \tan Z_1 \hat{A} I_1$

$$\therefore r_1 = s \tan \frac{A}{2}$$

$$\text{Also } r_1 = I_1 Z_1$$

$$= BZ_1 \tan Z_1 \hat{B} I_1$$

$$= (AZ_1 - AB) \tan Z_1 \hat{B} I_1$$

$$= (s - c) \tan \frac{\pi - B}{2}$$

$$= (s - c) \cot \frac{B}{2},$$

$$\text{and similarly } r_1 = (s - b) \cot \frac{C}{2}.$$

$$(iii) a = BC$$

$$= BX_1 + X_1 C$$

$$= I_1 X_1 \cot X_1 \hat{B} I_1 + I_1 X_1 \cot X_1 \hat{C} I_1$$

$$= r_1 \left(\cot \frac{\pi - B}{2} + \cot \frac{\pi - C}{2} \right)$$

$$= r_1 \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)$$

$$= r_1 \frac{\sin \frac{B}{2} \cos \frac{C}{2} + \sin \frac{C}{2} \cos \frac{B}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= r_1 \frac{\sin \frac{B + C}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= r_1 \frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}}$$

$$\therefore r_1 = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$$

$$= \frac{2R \sin A \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$$

$$= 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

Note. In the same way can be proved the following identities concerning the remaining excircles :

$$r_2 = \frac{\Delta}{s - b} = s \tan \frac{B}{2} = (s - a) \cot \frac{C}{2} = (s - c) \cot \frac{A}{2} = 4R \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2}$$

$$r_3 = \frac{\Delta}{s-c} = s \tan \frac{C}{2} = (s-b) \cot \frac{A}{2} = (s-a) \cot \frac{B}{2} = 4R \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2}$$

§ 61. To determine the sides and angles of the pedal triangle.

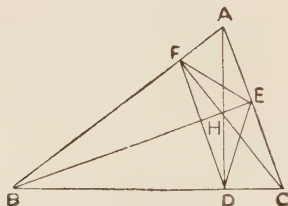


FIG. 79

In Fig. 79, (i) $\hat{BFC} = \hat{BEC}$

$\therefore$ B, F, E, C are concyclic and $\hat{AFE} = \hat{C}$

$\therefore$ triangles AEF, ABC are equiangular

$$\therefore \frac{EF}{BC} = \frac{AE}{AB}$$

$$\text{i.e., } \frac{EF}{a} = \cos A$$

$$\therefore EF = a \cos A = R \sin 2A$$

$$\text{Similarly } FD = b \cos B = R \sin 2B$$

$$\text{and } DE = c \cos C = R \sin 2C.$$

(ii) B, D, E, A are concyclic ($\hat{BDA} = \hat{BEA}$)

$$\therefore \hat{EDC} = \hat{A}$$

$$\text{Similarly } \hat{FDB} = \hat{A}$$

$$\text{But } \hat{EDF} = \pi - \hat{EDC} - \hat{FDB}$$

$$\therefore \hat{EDF} = \pi - 2A$$

$$\text{Similarly } \hat{FED} = \pi - 2B$$

$$\text{and } \hat{DFE} = \pi - 2C.$$

Example 1. Prove that $\Delta = \frac{(b+c)rr_1}{r+r_1}$.

$$\frac{(b+c)rr_1}{r+r_1} = \frac{(b+c) \frac{\Delta}{s} \cdot \frac{\Delta}{s-a}}{\frac{\Delta}{s} + \frac{\Delta}{s-a}}$$

$$\begin{aligned}
 & \frac{(b+c)\Delta^2}{s(s-a)} \\
 &= \frac{(2s-a)\Delta}{s(s-a)} \\
 &= \frac{(b+c)\Delta}{(2s-a)} \\
 &= \Delta.
 \end{aligned}$$

Example 2. Prove that the area of the pedal triangle of the triangle ABC is

$$\frac{1}{2}R^2 \sin 2A \sin 2B \sin 2C.$$

In Fig. 79, $EF = R \sin 2A$, $FD = R \sin 2B$, and $\angle BFE = \pi - 2C$
 $\therefore$ area of pedal triangle $= \frac{1}{2}R \sin 2A \cdot R \sin 2B \cdot \sin (\pi - 2C)$
 $= \frac{1}{2}R^2 \sin 2A \sin 2B \sin 2C.$

Example 3. Prove that (i) $\angle I_3 I_1 I_2 = \frac{\pi - A}{2}$

$$(ii) \quad I_2 I_3 = 4R \cos \frac{A}{2}.$$

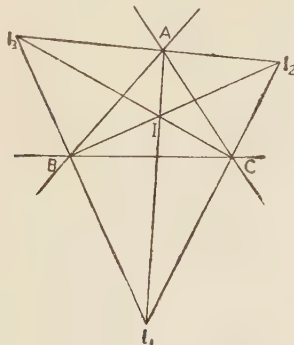


FIG. 80

$$\begin{aligned}
 \text{In Fig. 80, (i) } \angle I_3 I_1 I_2 &= \pi - \angle I_1 B C - \angle I_1 C B \\
 &= \pi - \frac{\pi - B}{2} - \frac{\pi - C}{2} \\
 &= \frac{B + C}{2} \\
 &= \frac{\pi - A}{2}.
 \end{aligned}$$

(ii) AI_1 and AI_2 are the internal and external bisectors of $\hat{A}$

$\therefore AI_1$ is perpendicular to AI_2

$\therefore AI_1$ is a perpendicular of the triangle $I_1I_2I_3$

Similarly BI_2 and CI_3 are perpendiculars of the triangle $I_1I_2I_3$

$\therefore \triangle ABC$ is the pedal triangle of $\triangle I_1I_2I_3$

$$\therefore BC = I_2I_3 \cos \hat{I_3I_1I_2}$$

$$\therefore a = I_2I_3 \cos \frac{\pi - A}{2}$$

$$= I_2I_3 \sin \frac{A}{2}$$

$$\therefore I_2I_3 = \frac{a}{\sin \frac{A}{2}}$$

$$= 4R \cos \frac{A}{2}.$$

Example 4. Prove that $IH^2 = 2r^2 - 4R^2 \cos A \cos B \cos C$.

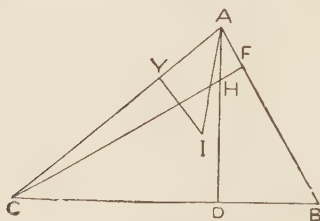


FIG. 81

$$I\hat{A}H \text{ (Fig. 81)} = C\hat{A}H - C\hat{A}I$$

$$= \left(\frac{\pi}{2} - C \right) - \frac{A}{2}$$

$$= \frac{A + B + C}{2} - C - \frac{A}{2}$$

$$= \frac{B - C}{2}$$

$$IA = \frac{IY}{\sin Y\hat{A}I}$$

$$= \frac{r}{\sin \frac{A}{2}}$$

$$= 4R \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\begin{aligned}
 AH &= \frac{AF}{\cos \hat{FAH}} \\
 &= \frac{AC \cos \hat{FAC}}{\cos \hat{FAH}} \\
 &= \frac{b \cos A}{\cos \left(\frac{\pi}{2} - B \right)} \\
 &= \frac{b \cos A}{\sin B} \\
 &= 2R \cos A
 \end{aligned}$$

$$\text{But } IH^2 = IA^2 + AH^2 - 2IA \cdot AH \cos \hat{IAH}$$

$$\begin{aligned}
 \therefore &= 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} + 4R^2 \cos^2 A \\
 &\quad - 16R^2 \sin \frac{B}{2} \sin \frac{C}{2} \cos A \cos \frac{B-C}{2} \\
 &= 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} + 4R^2 \cos^2 A \\
 &\quad - 16R^2 \sin \frac{B}{2} \sin \frac{C}{2} \cos A \left(\cos \frac{B}{2} \cos \frac{C}{2} + \sin \frac{B}{2} \sin \frac{C}{2} \right) \\
 &= 16R^2 \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} (1 - \cos A) + 4R^2 \cos^2 A \\
 &\quad - 4R^2 \sin B \sin C \cos A \\
 &= 32R^2 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} - 4R^2 \cos A (\sin B \sin C - \cos A) \\
 &= 2r^2 - 4R^2 \cos A (\sin B \sin C + \cos \overline{B+C}) \\
 &= 2r^2 - 4R^2 \cos A \cos B \cos C.
 \end{aligned}$$

Example 5. Prove that $\sin B \hat{A}A' = \frac{a \sin B}{\sqrt{2b^2 + 2c^2 - a^2}}$, A' being the mid-point of BC .

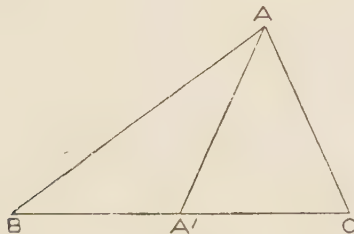


FIG. 82

Since AA' (Fig. 82) is a median of $\triangle ABC$, $AC^2 + AB^2 = 2BA'^2 + 2AA'^2$

$$\text{i.e., } b^2 + c^2 = \frac{a^2}{2} + 2AA'^2$$

$$\text{whence } AA' = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$$

$$\text{But } \frac{\sin \hat{BAA'}}{BA'} = \frac{\sin B}{AA'}$$

$$\therefore \frac{\sin \hat{BAA'}}{\frac{a}{2}} = \frac{\sin B}{\frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}}$$

$$\therefore \sin \hat{BAA'} = \frac{a \sin B}{\sqrt{2b^2 + 2c^2 - a^2}}.$$

EXERCISES

Prove that

$$1. r = r_1 \tan \frac{B}{2} \tan \frac{C}{2}.$$

$$2. rr_1 = r_2 r_3 \tan^2 \frac{A}{2}.$$

$$3. c = r_2 \tan \frac{C}{2} + r_3 \tan \frac{A}{2}.$$

$$4. r_1 \left(\cot \frac{A}{2} - \tan \frac{B}{2} \right) = r_2 \tan \frac{C}{2} + r_3 \tan \frac{A}{2}.$$

$$5. \Delta = r \left(r_1 \tan \frac{B}{2} + r_2 \tan \frac{C}{2} + r_3 \tan \frac{A}{2} \right).$$

$$6. \Delta = \frac{ar_2 r_3}{r_2 + r_3}.$$

$$8. \Delta = \frac{(b-c)r_2 r_3}{r_2 - r_3}.$$

$$7. \Delta = \frac{arr_1}{r_1 - r}.$$

$$9. \Delta = \sqrt{rr_1 r_2 r_3}.$$

$$10. r_1 r_2 r_3 = s \Delta.$$

$$11. rr_2 r_3 = (s-a) \Delta.$$

$$12. 4R\Delta = a(rr_1 + r_2 r_3) = b(rr_2 + r_3 r_1) = c(rr_3 + r_1 r_2).$$

$$13. \Delta = Rr(\sin A + \sin B + \sin C).$$

$$14. \Delta = Rr_3(\sin A + \sin B - \sin C).$$

$$15. \Delta = \frac{r^2(r_2 - r_3)(r_3 - r_1)(r_1 - r_2)}{(b-c)(c-a)(a-b)}.$$

$$19. \tan A = \frac{2\sqrt{rr_1 r_2 r_3}}{r_2 r_3 - rr_1}.$$

$$16. abc = 4Rrs.$$

$$17. \cos \frac{A}{2} = \sqrt{\frac{r_2 r_3}{bc}}.$$

$$20. \sin A = \frac{2\sqrt{rr_1 r_2 r_3}}{rr_1 + r_2 r_3}.$$

$$18. \sin \frac{A}{2} = \sqrt{\frac{rr_1}{bc}}.$$

$$21. \cos A = \frac{r_2 r_3 - rr_1}{r_2 r_3 + rr_1}.$$

22. $A = 90^\circ$ if $r_1 : r_2 = r_3 : r$.
23. $a = b$ if $r_1 = r_2$.
24. $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$.
25. $\frac{1}{r} = \frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}$.
26. $s = \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$.
27. $r_1 = \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2} \tan \frac{A}{2}$.
28. $\Delta = r \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$.
29. $b^2 + c^2 - a^2 = 2(r_2 r_3 - r r_1)$.
30. $4R = r_1 + r_2 + r_3 - r$.
31. $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$.
32. $(r_2 - r_3) \tan \frac{A}{2} + (r_3 - r_1) \tan \frac{B}{2} + (r_1 - r_2) \tan \frac{C}{2} = 0$.
33. $\frac{1}{s} = \frac{1}{a} \left(1 - \frac{r}{r_1}\right) = \frac{1}{b} \left(1 - \frac{r}{r_2}\right) = \frac{1}{c} \left(1 - \frac{r}{r_3}\right)$.
34. $\left(1 - \frac{r}{r_1}\right) \left(1 - \frac{r}{r_2}\right) \left(1 - \frac{r}{r_3}\right) = \frac{4Rr}{r_2 r_3 + r_3 r_1 + r_1 r_2}$.
35. $\left(\frac{r_1}{r} - 1\right) \left(\frac{r_2}{r} - 1\right) \left(\frac{r_3}{r} - 1\right) = \frac{4R}{r}$.
36. $r(r_1 - r_2)(r_2 - r_3)(r_3 - r_1) = (a - b)(b - c)(c - a) \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$.
37. $A = 45^\circ$ if $HA = BC$.
38. $HD = 2R \cos B \cos C$.
39. Circumradius of pedal triangle $= \frac{R}{2}$.
40. Inradius of pedal triangle $= 2R \cos A \cos B \cos C$.
41. Exradii of pedal triangle have lengths $2R \cos A \sin B \sin C$, $2R \cos B \sin C \sin A$, $2R \cos C \sin A \sin B$.
42. Area ΔABC : area of pedal triangle $= \frac{1}{2} \sec A \sec B \sec C$
43. Perimeter of ΔABC : perimeter of pedal triangle
$$= \frac{1}{4} \operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2}$$
44. $AD^2 = \frac{2\Delta}{\cot B + \cot C}$.
45. Circumradius of $\Delta I_1 I_2 I_3 = 2R$.
46. Area of $\Delta I_1 I_2 I_3 = 8R^2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$.
47. Perimeter of the triangle $I_1 I_2 I_3 = 16R \cos \frac{\pi - A}{4} \cos \frac{\pi - B}{4} \cos \frac{\pi - C}{4}$.
48. $Y_1 Z_1 : Z_1 X_1 : X_1 Y_1 = \cos \frac{A}{2} : \sin \frac{B}{2} : \sin \frac{C}{2}$.
49. $AI_1 = 4R \cos \frac{B}{2} \cos \frac{C}{2}$.
50. $AI_2 \cdot AI_3 = AI \cdot AI_1 = bc$.
51. $II_1 = 4R \sin \frac{A}{2}$.

$$52. AI \cdot BI \cdot CI = 4Rr^2.$$

$$53. AI^2 + AI_1^2 + AI_2^2 + AI_3^2 = 16R^2.$$

$$54. a \cot A + b \cot B + c \cot C = 2(R + r).$$

$$55. OH = R \sqrt{1 - 8 \cos A \cos B \cos C}.$$

$$56. OI = R \sqrt{1 - 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \text{ and hence } OI^2 = R^2 - 2Rr.$$

$$57. OI_1 = R \sqrt{1 + 8 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} \text{ and hence } OI_1^2 = R^2 + 2Rr_1.$$

$$58. A'D = \frac{c^2 - b^2}{2a} \text{ and } \cot AA'C = \frac{c^2 - b^2}{4\Delta}, A' \text{ being the mid-point of } BC$$

$$59. \sin AA'C = \frac{2b \sin c}{\sqrt{2b^2 + 2c^2 - a^2}}, A' \text{ being the mid-point of } BC.$$

$$60. \cot BAA' = 2 \cot A + \cot B, A' \text{ being the mid-point of } BC.$$

$$61. AU = \frac{2\Delta}{(b+c) \sin \frac{A}{2}}, \text{ if } AI \text{ produced meets } BC \text{ in } U.$$

APPENDIX

I. PROJECTION

§ 1. NOTATION

Let OX be any straight line and A and B any two points on it ; if we take AB to be positive or negative according as AB has the same direction as or the opposite direction to OX , then $AB = -BA$. When we thus take into account the direction of AB we shall speak of the segment AB .

Note 1. We have considered AB to be part of the straight line OX ; but, irrespective of any reference to OX , we shall speak of the segment AB , implying that $AB = -BA$.

Note 2. If OX is the positive direction of a straight line, *e.g.*, the x -axis, then AB is a positive segment when its direction is that of OX . AB is a negative segment when the direction of BA is that of OX .

With the preceding convention, we note that if A , B , and C are any three points on a straight line, $AC = AB + BC$.

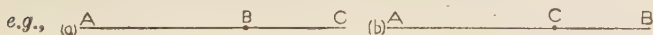


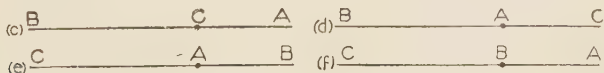
FIG. 1

In Fig. 1 (a), $AC = AB + BC$

(b), $AC = AB - CB$

$= AB + BC$ ($BC = -CB$).

The student should satisfy himself that this relation is true in the four remaining cases :



§ 2. DEFINITION OF PROJECTION

The projection of any segment AB (Fig. 2) on another line OX is the segment A_1B_1 , where A_1 and B_1 are the feet of the perpendiculars from A and B to OX .

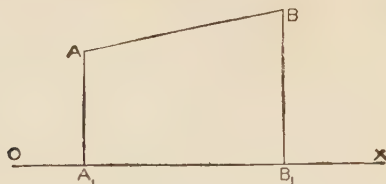


FIG. 2

For conciseness we shall use the notation $p(AB)$ to stand for the projection of AB .

§ 3. *Theorem I.* Projection of AB on $OX = -$ projection of BA on OX .

$$\begin{aligned}\text{On } OX \text{ (Fig. 2), } p(AB) &= A_1B_1 \\ &= -B_1A_1 \\ &= -p(BA).\end{aligned}$$

Theorem II. If A , B , and C are the vertices of a triangle, then the projection of AC on OX is equal to the sum of the projections of AB and BC on OX .

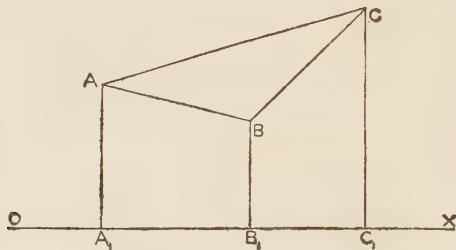


FIG. 3

Let A_1 , B_1 , and C_1 (Fig. 3) be the projections of A , B , and C on OX .

$$\begin{aligned}\text{Then } A_1C_1 &= A_1B_1 + B_1C_1 \\ \therefore \text{ on } OX, p(AC) &= p(AB) + p(BC).\end{aligned}$$

Note. The student should satisfy himself that this theorem is true for all relative positions of A_1 , B_1 , and C_1 (see the six cases of § 1).

§ 4. DEFINITION OF EQUIVALENT SEGMENTS. Two segments AB , PQ , are said to be equivalent when they are equal in length and are collinear or parallel, B being on the same side of A as Q is of P .

Theorem III. The projections on any line OX of equivalent segments AB , PQ , are equivalent segments.

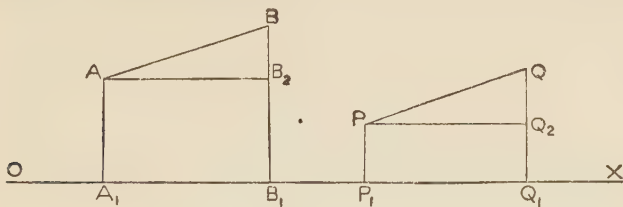


FIG. 4

Let AA_1 , BB_1 , PP_1 , and QQ_1 (Fig. 4) be perpendicular to OX , and let AB_2 and PQ_2 be parallel to OX .

Then $\triangle ABB_2 \equiv \triangle PQQ_2$

$\therefore AB_2$, PQ_2 are equal in length

$\therefore A_1B_1$, P_1Q_1 " "

but A_1B_1 , P_1Q_1 are collinear

and $\therefore B$ is on the same side of A as Q is of P

B_1 " " " A_1 " Q_1 " P_1

$\therefore A_1B_1$, P_1Q_1 are equivalent segments.

Theorem IV. If AB is a segment (+ive or -ive) of a straight line whose positive direction makes an angle θ with OX , then (1) projection of AB on $OX = AB \cos \theta$, (2) projection of AB on $OY = AB \sin \theta$.

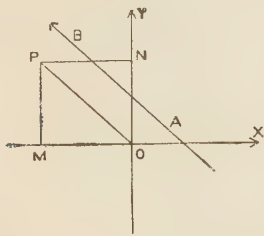


FIG. 5

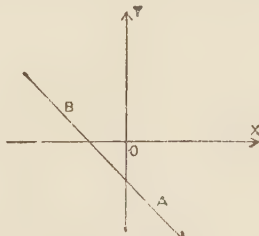


FIG. 6

Case I. Let AB (Fig. 5) be a +ive segment of length r ; let OP be an equivalent segment, P the point (x, y) , and M and N the projections of P on OX and OY .

$$\begin{aligned}
 \text{Then on } OX, p(AB) &= p(OP) \\
 &= OM \\
 &= r \cdot \frac{x}{r} \\
 &= r \cos \theta \\
 &= AB \cos \theta
 \end{aligned}$$

$$\begin{aligned}
 \text{On } OY, p(AB) &= p(OP) \\
 &= ON \\
 &= r \cdot \frac{y}{r} \\
 &= r \sin \theta \\
 &= AB \sin \theta.
 \end{aligned}$$

Case II. Let AB (Fig. 6) be a -ive segment. Then segment BA is +ive.

$$\therefore \text{ on } OX, p(BA) = BA \cos \theta \text{ (Case I)}$$

$$\therefore p(AB) = AB \cos \theta$$

$$\text{Similarly on } OY, p(AB) = AB \sin \theta.$$

II. PROOF, BY PROJECTION, OF THE FOLLOWING RELATIONS:

$$(1) \cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$(2) \sin(A+B) = \sin A \cos B + \cos A \sin B.$$

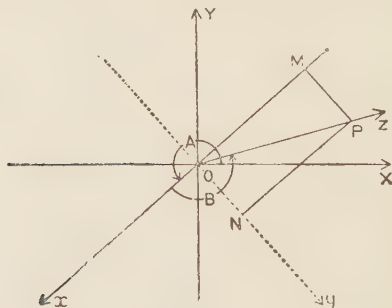


FIG. 7

Let a line of length r (Fig. 7) start from OX and turn through an angle A to the direction Ox : let it turn through a further angle B

till it reaches the direction Oz and takes up the position OP , where P is the point (x, y) .

Then OP has described the angle $(A + B)$ from OX .

$\therefore \frac{x}{r} = \cos(A + B)$ and $\frac{y}{r} = \sin(A + B)$, by definition of circular functions

$$\begin{aligned} \therefore \text{on } x\text{-axis, } p(OP) = x = r \cos(A + B) \\ \text{and on } y\text{-axis, } p(OP) = y = r \sin(A + B) \end{aligned} \quad (a)$$

Now let Ox be the positive direction of a new axis of abscissæ, and let Oy , making $+90^\circ$ with Ox , be the positive direction of the new axis of ordinates. Then Oy makes an angle $(A + 90^\circ)$ with OX . Let M and N be the projections of P on Ox and Oy .

Then, with reference to axes Ox, Oy ,

$$\begin{aligned} \frac{OM}{r} = \cos B \text{ and } \frac{ON}{r} = \sin B \\ \therefore OM = r \cos B \text{ and } ON = r \sin B \end{aligned} \quad (b)$$

Again, OM is a segment of a line whose +ive direction Ox makes angle A with OX

and ON is a segment of a line whose +ive direction Oy makes angle $A + 90^\circ$ with OX

also $p(OP) = p(OM) + p(MP)$

$= p(OM) + p(ON)$ (ON, MP are equivalent segments.)

$\therefore$ (1) taking projections on OX ,

$$\begin{aligned} r \cos(A + B) &= OM \cos A + ON \cos(A + 90^\circ), \text{ by } (a) \\ &= r \cos B \cos A + r \sin B (-\sin A), \text{ by } (b) \end{aligned}$$

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B$$

(2) taking projections on OY ,

$$\begin{aligned} r \sin(A + B) &= OM \sin A + ON \sin(A + 90^\circ), \text{ by } (a) \\ &= r \cos B \sin A + r \sin B \cos A, \text{ by } (b) \end{aligned}$$

$$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B.$$

Relations (1) and (2) are true for all values of A and B , hence substituting $(-B)$ for B we have :

$$\begin{aligned} \text{from (1), } \cos(A - B) &= \cos A \cos(-B) - \sin A \sin(-B) \\ &= \cos A \cos B + \sin A \sin B \end{aligned}$$

$$\begin{aligned} \text{from (2), } \sin(A - B) &= \sin A \cos(-B) + \cos A \sin(-B) \\ &= \sin A \cos B - \cos A \sin B. \end{aligned}$$

TABLES OF LOGARITHMS, CIRCULAR
FUNCTIONS, ETC.

LOGARITHMS

											Differences									
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37	
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34	
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31	
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29	
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27	
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25	
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24	
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22	
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21	
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20	
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19	
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18	
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17	
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17	
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16	
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15	
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15	
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14	
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14	
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13	
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13	
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12	
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12	
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12	
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11	
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11	
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11	
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10	
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10	
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10	
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10	
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9	
42	6232	6243	6253	6263	6271	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9	
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9	
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9	
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9	
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8	
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8	
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8	
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8	
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8	
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8	
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7	
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7	
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7	

LOGARITHMS

											Differences									
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9	
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7	
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7	
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7	
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7	
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7	
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6	
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6	
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6	
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6	
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6	
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6	
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6	
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6	
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6	
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6	
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6	
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5	
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5	
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5	
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5	
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5	
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5	
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	5	5	
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5	
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5	
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5	
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5	
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5	
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5	
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5	
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5	
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5	
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4	
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4	
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4	
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4	
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4	
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4	
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4	
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4	
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4	
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4	
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4	
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4	
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	4	4	

ANTILOGARITHMS

											Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
·01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
·02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
·03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
·04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
·05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
·06	1148	1151	1153	1156	1159	1161	1164	1167	1169	1172	0	1	1	1	1	2	2	2	2
·07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
·08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
·09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
·10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3
·11	1288	1291	1294	1297	1300	1303	1306	1309	1312	1315	0	1	1	1	1	2	2	2	3
·12	1318	1321	1324	1327	1330	1334	1337	1340	1343	1346	0	1	1	1	1	2	2	2	3
·13	1349	1352	1355	1358	1361	1365	1368	1371	1374	1377	0	1	1	1	1	2	2	2	3
·14	1380	1384	1387	1390	1393	1396	1400	1403	1406	1409	0	1	1	1	1	2	2	2	3
·15	1413	1416	1419	1422	1426	1429	1432	1435	1439	1442	0	1	1	1	1	2	2	2	3
·16	1445	1449	1452	1455	1459	1462	1466	1469	1472	1476	0	1	1	1	1	2	2	2	3
·17	1479	1483	1486	1489	1493	1496	1500	1503	1507	1510	0	1	1	1	1	2	2	2	3
·18	1514	1517	1521	1524	1528	1531	1535	1538	1542	1545	0	1	1	1	1	2	2	2	3
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	1	2	2	2	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	1	2	2	2	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	1	1	2	2	2	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	1	1	2	2	2	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	1	1	2	2	2	3
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	1	1	2	2	2	3
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	1	1	2	2	2	3
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	1	1	2	2	2	3
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	1	1	2	2	2	3
·28	1905	1910	1914	1919	1923	1928	1932	1936	1941	1945	0	1	1	1	1	2	2	2	3
·29	1950	1954	1959	1963	1968	1972	1977	1982	1986	1991	0	1	1	1	1	2	2	2	3
·30	1995	2000	2004	2009	2014	2018	2023	2028	2032	2037	0	1	1	1	1	2	2	2	3
·31	2042	2046	2051	2056	2061	2065	2070	2075	2080	2084	0	1	1	1	1	2	2	2	3
·32	2089	2094	2099	2104	2109	2113	2118	2123	2128	2133	0	1	1	1	1	2	2	2	3
·33	2138	2143	2148	2153	2158	2163	2168	2173	2178	2183	0	1	1	1	1	2	2	2	3
·34	2188	2193	2198	2203	2208	2213	2218	2223	2228	2234	1	1	2	2	2	3	3	4	4
·35	2239	2244	2249	2254	2259	2265	2270	2275	2280	2286	1	1	2	2	2	3	3	4	4
·36	2291	2296	2301	2307	2312	2317	2323	2328	2333	2339	1	1	2	2	2	3	3	4	4
·37	2344	2350	2355	2360	2366	2371	2377	2382	2388	2393	1	1	2	2	2	3	3	4	4
·38	2399	2404	2410	2415	2421	2427	2432	2438	2443	2449	1	1	2	2	2	3	3	4	4
·39	2455	2460	2466	2472	2477	2483	2489	2495	2500	2506	1	1	2	2	2	3	3	4	4
·40	2512	2518	2523	2529	2535	2541	2547	2553	2559	2564	1	1	2	2	2	3	3	4	4
·41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	2	3	3	4	4
·42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	2	3	3	4	4
·43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	2	2	3	3	4	4
·44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	2	2	3	3	4	4
·45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	2	2	3	3	4	4
·46	2884	2891	2897	2904	2911	2917	2924	2931	2938	2944	1	1	2	2	2	3	3	4	4
·47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	2	2	3	3	4	4
·48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	2	2	3	3	4	4
·49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	2	2	3	3	4	4

ANTILOGARITHMS

											Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
-50	3162	3170	3177	3184	3192	3199	3206	3214	3221	3228	1	1	2	3	4	4	5	6	7
-51	3236	3243	3251	3258	3266	3273	3281	3289	3296	3304	1	2	2	3	4	5	5	6	7
-52	3311	3319	3327	3334	3342	3350	3357	3365	3373	3381	1	2	2	3	4	5	5	6	7
-53	3388	3396	3404	3412	3420	3428	3436	3443	3451	3459	1	2	2	3	4	5	6	6	7
-54	3467	3475	3483	3491	3499	3508	3516	3524	3532	3540	1	2	2	3	4	5	6	6	7
-55	3548	3556	3565	3573	3581	3589	3597	3606	3614	3622	1	2	2	3	4	5	6	7	7
-56	3631	3639	3648	3656	3664	3673	3681	3690	3698	3707	1	2	3	3	4	5	6	7	8
-57	3715	3724	3733	3741	3750	3758	3767	3776	3784	3793	1	2	3	3	4	5	6	7	8
-58	3802	3811	3819	3828	3837	3846	3855	3864	3873	3882	1	2	3	4	4	5	6	7	8
-59	3890	3899	3908	3917	3926	3936	3945	3954	3963	3972	1	2	3	4	5	5	6	7	8
-60	3981	3990	3999	4009	4018	4027	4036	4046	4055	4064	1	2	3	4	5	6	6	7	8
-61	4074	4083	4093	4102	4111	4121	4130	4140	4150	4159	1	2	3	4	5	6	7	8	9
-62	4169	4178	4188	4198	4207	4217	4227	4236	4246	4256	1	2	3	4	5	6	7	8	9
-63	4266	4276	4285	4295	4305	4315	4325	4335	4345	4355	1	2	3	4	5	6	7	8	9
-64	4365	4375	4385	4395	4406	4416	4426	4436	4446	4457	1	2	3	4	5	6	7	8	9
-65	4467	4477	4487	4498	4508	4519	4529	4539	4550	4560	1	2	3	4	5	6	7	8	9
-66	4571	4581	4592	4603	4613	4624	4634	4645	4656	4667	1	2	3	4	5	6	7	9	10
-67	4677	4688	4699	4710	4721	4732	4742	4753	4764	4775	1	2	3	4	5	7	8	9	10
-68	4786	4797	4808	4819	4831	4842	4853	4864	4875	4887	1	2	3	4	6	7	8	9	10
-69	4898	4909	4920	4932	4943	4955	4966	4977	4989	5000	1	2	3	5	6	7	8	9	10
-70	5012	5023	5035	5047	5058	5070	5082	5093	5105	5117	1	2	4	5	6	7	8	9	11
-71	5129	5140	5152	5164	5176	5188	5200	5212	5224	5236	1	2	4	5	6	7	8	10	11
-72	5248	5260	5272	5284	5297	5309	5321	5333	5346	5358	1	2	4	5	6	7	9	10	11
-73	5370	5383	5395	5408	5420	5433	5445	5458	5470	5483	1	3	4	5	6	8	9	10	11
-74	5495	5508	5521	5534	5546	5559	5572	5585	5598	5610	1	3	4	5	6	8	9	10	12
-75	5623	5636	5649	5662	5675	5689	5702	5715	5728	5741	1	3	4	5	7	8	9	10	12
-76	5754	5768	5781	5794	5808	5821	5834	5848	5861	5875	1	3	4	5	7	8	9	11	12
-77	5888	5902	5916	5929	5943	5957	5970	5984	5998	6012	1	3	4	5	7	8	10	11	12
-78	6026	6039	6053	6067	6081	6095	6109	6124	6138	6152	1	3	4	6	7	8	10	11	13
-79	6166	6180	6194	6209	6223	6237	6252	6266	6281	6295	1	3	4	6	7	9	10	11	13
-80	6310	6324	6339	6353	6368	6383	6397	6412	6427	6442	1	3	4	6	7	9	10	12	13
-81	6457	6471	6486	6501	6516	6531	6546	6561	6577	6592	2	3	5	6	8	9	11	12	14
-82	6607	6622	6637	6653	6668	6683	6699	6714	6730	6745	2	3	5	6	8	9	11	12	14
-83	6761	6776	6792	6808	6823	6839	6855	6871	6887	6902	2	3	5	6	8	9	11	13	14
-84	6918	6934	6950	6966	6982	6998	7015	7031	7047	7063	2	3	5	6	8	10	11	13	15
-85	7079	7096	7112	7129	7145	7161	7178	7194	7211	7228	2	3	5	7	8	10	12	13	15
-86	7244	7261	7278	7295	7311	7328	7345	7362	7379	7396	2	3	5	7	8	10	12	13	15
-87	7413	7430	7447	7464	7482	7499	7516	7534	7551	7568	2	3	5	7	9	10	12	14	16
-88	7586	7603	7621	7638	7656	7674	7691	7709	7727	7745	2	4	5	7	9	11	12	14	16
-89	7762	7780	7798	7816	7834	7852	7870	7889	7907	7925	2	4	5	7	9	11	13	14	16
-90	7943	7962	7980	7998	8017	8035	8054	8072	8091	8110	2	4	6	7	9	11	13	15	17
-91	8128	8147	8166	8185	8204	8222	8241	8260	8279	8299	2	4	6	8	9	11	13	15	17
-92	8318	8337	8356	8375	8395	8414	8433	8453	8472	8492	2	4	6	8	10	12	14	15	17
-93	8511	8531	8551	8570	8590	8610	8630	8650	8670	8690	2	4	6	8	10	12	14	16	18
-94	8710	8730	8750	8770	8790	8810	8831	8851	8872	8892	2	4	6	8	10	12	14	16	18
-95	8913	8933	8954	8974	8995	9016	9036	9057	9078	9099	2	4	6	8	10	12	15	17	19
-96	9120	9141	9162	9183	9204	9226	9247	9268	9290	9311	2	4	6	8	11	13	15	17	19
-97	9333	9354	9376	9397	9419	9441	9462	9484	9506	9528	2	4	7	9	11	13	15	17	20
-98	9550	9572	9594	9616	9638	9661	9683	9705	9727	9750	2	4	7	9	11	13	16	18	20
-99	9772	9795	9817	9840	9863	9886	9908	9931	9954	9977	2	5	7	9	11	14	16	18	20

NATURAL SINES

											Differences				
	O'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0366	0384	0401	0419	0436	0454	0471	0488	0506	3	6	9	12	15
3	·0523	0541	0558	0576	0593	0610	0628	0645	0663	0680	3	6	9	12	15
4	·0698	0715	0732	0750	0767	0785	0802	0819	0837	0854	3	6	9	12	14
5	·0872	0889	0906	0924	0941	0958	0976	0993	1011	1028	3	6	9	12	14
6	·1045	1063	1080	1097	1115	1132	1149	1167	1184	1201	3	6	9	12	14
7	·1219	1236	1253	1271	1288	1305	1323	1340	1357	1374	3	6	9	12	14
8	·1392	1409	1426	1444	1461	1478	1495	1513	1530	1547	3	6	9	12	14
9	·1564	1582	1599	1616	1633	1650	1668	1685	1702	1719	3	6	9	12	14
10	·1736	1754	1771	1788	1805	1822	1840	1857	1874	1891	3	6	9	12	14
11	·1908	1925	1942	1959	1977	1994	2011	2028	2045	2062	3	6	9	11	14
12	·2079	2096	2113	2130	2147	2164	2181	2198	2215	2233	3	6	9	11	14
13	·2250	2267	2284	2300	2317	2334	2351	2368	2385	2402	3	6	8	11	14
14	·2419	2436	2453	2470	2487	2504	2521	2538	2554	2571	3	6	8	11	14
15	·2588	2605	2622	2639	2656	2672	2689	2706	2723	2740	3	6	8	11	14
16	·2756	2773	2790	2807	2823	2840	2857	2874	2890	2907	3	6	8	11	14
17	·2924	2940	2957	2974	2990	3007	3024	3040	3057	3074	3	6	8	11	14
18	·3090	3107	3123	3140	3156	3173	3190	3206	3223	3239	3	6	8	11	14
19	·3256	3272	3289	3305	3322	3338	3355	3371	3387	3404	3	5	8	11	14
20	·3420	3437	3453	3469	3486	3502	3518	3535	3551	3567	3	5	8	11	14
21	·3584	3600	3616	3633	3649	3665	3681	3697	3714	3730	3	5	8	11	14
22	·3746	3762	3778	3795	3811	3827	3843	3859	3875	3891	3	5	8	11	14
23	·3907	3923	3939	3955	3971	3987	4003	4019	4035	4051	3	5	8	11	13
24	·4067	4083	4099	4115	4131	4147	4163	4179	4195	4210	3	5	8	11	13
25	·4226	4242	4258	4274	4289	4305	4321	4337	4352	4368	3	5	8	11	13
26	·4384	4399	4415	4431	4446	4462	4478	4493	4509	4524	3	5	8	10	13
27	·4540	4555	4571	4586	4602	4617	4633	4648	4664	4679	3	5	8	10	13
28	·4695	4710	4726	4741	4756	4772	4787	4802	4818	4833	3	5	8	10	13
29	·4848	4863	4879	4894	4909	4924	4939	4955	4970	4985	3	5	8	10	13
30	·5000	5015	5030	5045	5060	5075	5090	5105	5120	5135	3	5	8	10	13
31	·5150	5165	5180	5195	5210	5225	5240	5255	5270	5284	2	5	7	10	12
32	·5299	5314	5329	5344	5358	5373	5388	5402	5417	5432	2	5	7	10	12
33	·5446	5461	5476	5490	5505	5519	5534	5548	5563	5577	2	5	7	10	12
34	·5592	5606	5621	5635	5650	5664	5678	5693	5707	5721	2	5	7	10	12
35	·5736	5750	5764	5779	5793	5807	5821	5835	5850	5864	2	5	7	9	12
36	·5878	5892	5906	5920	5934	5948	5962	5976	5990	6004	2	5	7	9	12
37	·6018	6032	6046	6060	6074	6088	6101	6115	6129	6143	2	5	7	9	12
38	·6157	6170	6184	6198	6211	6225	6239	6252	6266	6280	2	5	7	9	11
39	·6293	6307	6320	6334	6347	6361	6374	6388	6401	6414	2	4	7	9	11
40	·6428	6441	6455	6468	6481	6494	6508	6521	6534	6547	2	4	7	9	11
41	·6561	6574	6587	6600	6613	6626	6639	6652	6665	6678	2	4	7	9	11
42	·6691	6704	6717	6730	6743	6756	6769	6782	6794	6807	2	4	6	9	11
43	·6820	6833	6845	6858	6871	6884	6896	6909	6921	6934	2	4	6	8	11
44	·6947	6959	6972	6984	6997	7009	7022	7034	7046	7059	2	4	6	8	10

NATURAL SINES

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
45°	·7071	7083	7096	7108	7120	7133	7145	7157	7169	7181	2	4	6	8	10
46	·7193	7206	7218	7230	7242	7254	7266	7278	7290	7302	2	4	6	8	10
47	·7314	7325	7337	7349	7361	7373	7385	7396	7408	7420	2	4	6	8	10
48	·7431	7443	7455	7466	7478	7490	7501	7513	7524	7536	2	4	6	8	10
49	·7547	7559	7570	7581	7593	7604	7615	7627	7638	7649	2	4	6	8	9
50	·7660	7672	7683	7694	7705	7716	7727	7738	7749	7760	2	4	6	7	9
51	·7771	7782	7793	7804	7815	7826	7837	7848	7859	7869	2	4	5	7	9
52	·7880	7891	7902	7912	7923	7934	7944	7955	7965	7976	2	4	5	7	9
53	·7986	7997	8007	8018	8028	8039	8049	8059	8070	8080	2	3	5	7	9
54	·8090	8100	8111	8121	8131	8141	8151	8161	8171	8181	2	3	5	7	8
55	·8192	8202	8211	8221	8231	8241	8251	8261	8271	8281	2	3	5	7	8
56	·8290	8300	8310	8320	8329	8339	8348	8358	8368	8377	2	3	5	6	8
57	·8387	8396	8406	8415	8425	8434	8443	8453	8462	8471	2	3	5	6	8
58	·8480	8490	8499	8508	8517	8526	8536	8545	8554	8563	2	3	5	6	8
59	·8572	8581	8590	8599	8607	8616	8625	8634	8643	8652	1	3	4	6	7
60	·8660	8669	8678	8686	8695	8704	8712	8721	8729	8738	1	3	4	6	7
61	·8746	8755	8763	8771	8780	8788	8796	8805	8813	8821	1	3	4	6	7
62	·8829	8838	8846	8854	8862	8870	8878	8886	8894	8902	1	3	4	5	7
63	·8910	8918	8926	8934	8942	8949	8957	8965	8973	8980	1	3	4	5	6
64	·8988	8996	9003	9011	9018	9026	9033	9041	9048	9056	1	3	4	5	6
65	·9063	9070	9078	9085	9092	9100	9107	9114	9121	9128	1	2	4	5	6
66	·9135	9143	9150	9157	9164	9171	9178	9184	9191	9198	1	2	3	5	6
67	·9205	9212	9219	9225	9232	9239	9245	9252	9259	9265	1	2	3	4	6
68	·9272	9278	9285	9291	9298	9304	9311	9317	9323	9330	1	2	3	4	5
69	·9336	9342	9348	9354	9361	9367	9373	9379	9385	9391	1	2	3	4	5
70	·9397	9403	9409	9415	9421	9426	9432	9438	9444	9449	1	2	3	4	5
71	·9455	9461	9466	9472	9478	9483	9489	9494	9500	9505	1	2	3	4	5
72	·9511	9516	9521	9527	9532	9537	9542	9548	9553	9558	1	2	3	3	4
73	·9563	9568	9573	9578	9583	9588	9593	9598	9603	9608	1	2	2	3	4
74	·9613	9617	9622	9627	9632	9636	9641	9646	9650	9655	1	2	2	3	4
75	·9659	9664	9668	9673	9677	9681	9686	9690	9694	9699	1	1	2	3	4
76	·9703	9707	9711	9715	9720	9724	9728	9732	9736	9740	1	1	2	3	3
77	·9744	9748	9751	9755	9759	9763	9767	9770	9774	9778	1	1	2	3	3
78	·9781	9785	9789	9792	9796	9799	9803	9806	9810	9813	1	1	2	2	3
79	·9816	9820	9823	9826	9829	9833	9836	9839	9842	9845	1	1	2	2	3
80	·9848	9851	9854	9857	9860	9863	9866	9869	9871	9874	0	1	1	2	2
81	·9877	9880	9882	9885	9888	9890	9893	9895	9898	9900	0	1	1	2	2
82	·9903	9905	9907	9910	9912	9914	9917	9919	9921	9923	0	1	1	2	2
83	·9925	9928	9930	9932	9934	9936	9938	9940	9942	9943	0	1	1	1	2
84	·9945	9947	9949	9951	9952	9954	9956	9957	9959	9960	0	1	1	1	2
85	·9962	9963	9965	9966	9968	9969	9971	9972	9973	9974	0	0	1	1	1
86	·9976	9977	9978	9979	9980	9981	9982	9983	9984	9985	0	0	1	1	1
87	·9986	9987	9988	9989	9990	9990	9991	9992	9993	9993	0	0	0	1	1
88	·9994	9995	9995	9996	9996	9997	9997	9997	9998	9998	0	0	0	0	0
89	·9998	9999	9999	9999	9999	0000	0000	0000	0000	0000	0	0	0	0	0

NATURAL COSINES

Subtract differences

	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	Differences				
											1'	2'	3'	4'	5'
0°	1.0000	0000	0000	0000	0000	0000	9999	9999	9999	9999	0	0	0	0	0
1	.9998	9998	9998	9997	9997	9997	9996	9996	9995	9995	0	0	0	0	0
2	.9994	9993	9993	9992	9991	9990	9990	9989	9988	9987	0	0	0	1	1
3	.9986	9985	9984	9983	9982	9981	9980	9979	9978	9977	0	0	1	1	1
4	.9976	9974	9973	9972	9971	9969	9968	9966	9965	9963	0	0	1	1	1
5	.9962	9960	9959	9957	9956	9954	9952	9951	9949	9947	0	1	1	1	2
6	.9945	9943	9942	9940	9938	9936	9934	9932	9930	9928	0	1	1	1	2
7	.9925	9923	9921	9919	9917	9914	9912	9910	9907	9905	0	1	1	2	2
8	.9903	9900	9898	9895	9893	9890	9888	9885	9882	9880	0	1	1	2	2
9	.9877	9874	9871	9869	9866	9863	9860	9857	9854	9851	0	1	1	2	2
10	.9848	9845	9842	9839	9836	9833	9829	9826	9823	9820	1	1	2	2	3
11	.9816	9813	9810	9806	9803	9799	9796	9792	9789	9785	1	1	2	2	3
12	.9781	9778	9774	9770	9767	9763	9759	9755	9751	9748	1	1	2	3	3
13	.9744	9740	9736	9732	9728	9724	9720	9715	9711	9707	1	1	2	3	3
14	.9703	9699	9694	9690	9686	9681	9677	9673	9668	9664	1	1	2	3	4
15	.9659	9655	9650	9646	9641	9636	9632	9627	9622	9617	1	2	2	3	4
16	.9613	9608	9603	9598	9593	9588	9583	9578	9573	9568	1	2	2	3	4
17	.9563	9558	9553	9548	9542	9537	9532	9527	9521	9516	1	2	3	3	4
18	.9511	9505	9500	9494	9489	9483	9478	9472	9466	9461	1	2	3	4	5
19	.9455	9449	9444	9438	9432	9426	9421	9415	9409	9403	1	2	3	4	5
20	.9397	9391	9385	9379	9373	9367	9361	9354	9348	9342	1	2	3	4	5
21	.9336	9330	9323	9317	9311	9304	9298	9291	9285	9278	1	2	3	4	5
22	.9272	9265	9259	9252	9245	9239	9232	9225	9219	9212	1	2	3	4	6
23	.9205	9198	9191	9184	9178	9171	9164	9157	9150	9143	1	2	3	5	6
24	.9135	9128	9121	9114	9107	9100	9092	9085	9078	9070	1	2	4	5	6
25	.9063	9056	9048	9041	9033	9026	9018	9011	9003	8996	1	3	4	5	6
26	.8988	8980	8973	8965	8957	8949	8942	8934	8926	8918	1	3	4	5	6
27	.8910	8902	8894	8886	8878	8870	8862	8854	8846	8838	1	3	4	5	7
28	.8829	8821	8813	8805	8796	8788	8780	8771	8763	8755	1	3	4	6	7
29	.8746	8738	8729	8721	8712	8704	8695	8686	8678	8669	1	3	4	6	7
30	.8660	8652	8643	8634	8625	8616	8607	8599	8590	8581	1	3	4	6	7
31	.8572	8563	8554	8545	8536	8526	8517	8508	8499	8490	2	3	5	6	8
32	.8480	8471	8462	8453	8443	8434	8425	8415	8406	8396	2	3	5	6	8
33	.8387	8377	8368	8358	8348	8339	8329	8320	8310	8300	2	3	5	6	8
34	.8290	8281	8271	8261	8251	8241	8231	8221	8211	8202	2	3	5	7	8
35	.8192	8181	8171	8161	8151	8141	8131	8121	8111	8100	2	3	5	7	8
36	.8090	8080	8070	8059	8049	8039	8028	8018	8007	7997	2	3	5	7	9
37	.7986	7976	7965	7955	7944	7934	7923	7912	7902	7891	2	4	5	7	9
38	.7880	7869	7859	7848	7837	7826	7815	7804	7793	7782	2	4	5	7	9
39	.7771	7760	7749	7738	7727	7716	7705	7694	7683	7672	2	4	6	7	9
40	.7660	7649	7638	7627	7615	7604	7593	7581	7570	7559	2	4	6	8	9
41	.7547	7536	7524	7513	7501	7490	7478	7466	7455	7443	2	4	6	8	10
42	.7431	7420	7408	7396	7385	7373	7361	7349	7337	7325	2	4	6	8	10
43	.7314	7302	7290	7278	7266	7254	7242	7230	7218	7206	2	4	6	8	10
44	.7193	7181	7169	7157	7145	7133	7120	7108	7096	7083	2	4	6	8	10

NATURAL COSINES

Subtract differences

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
45°	·7071	7059	7046	7034	7022	7009	6997	6984	6972	6959	2	4	6	8	10
46	·6947	6934	6921	6909	6896	6884	6871	6858	6845	6833	2	4	6	8	11
47	·6820	6807	6794	6782	6769	6756	6743	6730	6717	6704	2	4	6	9	11
48	·6691	6678	6665	6652	6639	6626	6613	6600	6587	6574	2	4	7	9	11
49	·6561	6547	6534	6521	6508	6494	6481	6468	6455	6441	2	4	7	9	11
50	·6128	6414	6401	6388	6374	6361	6347	6334	6320	6307	2	4	7	9	11
51	·6293	6280	6266	6252	6239	6225	6211	6198	6184	6170	2	5	7	9	11
52	·6157	6143	6129	6115	6101	6088	6074	6060	6046	6032	2	5	7	9	12
53	·6018	6004	5990	5976	5962	5948	5934	5920	5906	5892	2	5	7	9	12
54	·5878	5864	5850	5835	5821	5807	5793	5779	5764	5750	2	5	7	9	12
55	·5736	5721	5707	5693	5678	5664	5650	5635	5621	5606	2	5	7	10	12
56	·5592	5577	5563	5548	5534	5519	5505	5490	5476	5461	2	5	7	10	12
57	·5446	5432	5417	5402	5388	5373	5358	5344	5329	5314	2	5	7	10	12
58	·5299	5284	5270	5255	5240	5225	5210	5195	5180	5165	2	5	7	10	12
59	·5150	5135	5120	5105	5090	5075	5060	5045	5030	5015	3	5	8	10	13
60	·5000	4985	4970	4955	4939	4924	4909	4894	4879	4863	3	5	8	10	13
61	·4818	4833	4818	4802	4787	4772	4756	4741	4726	4710	3	5	8	10	13
62	·4695	4679	4664	4648	4633	4617	4602	4586	4571	4555	3	5	8	10	13
63	·4540	4524	4509	4493	4478	4462	4446	4431	4415	4399	3	5	8	10	13
64	·4384	4368	4352	4337	4321	4305	4289	4274	4258	4242	3	5	8	11	13
65	·4226	4210	4195	4179	4163	4147	4131	4115	4099	4083	3	5	8	11	13
66	·4067	4051	4035	4019	4003	3987	3971	3955	3939	3923	3	5	8	11	13
67	·3907	3891	3875	3859	3843	3827	3811	3795	3778	3762	3	5	8	11	13
68	·3746	3730	3714	3697	3681	3665	3649	3633	3616	3600	3	5	8	11	14
69	·3584	3567	3551	3535	3518	3502	3486	3469	3453	3437	3	5	8	11	14
70	·3420	3404	3387	3371	3355	3338	3322	3305	3289	3272	3	5	8	11	14
71	·3256	3239	3223	3206	3190	3173	3156	3140	3123	3107	3	6	8	11	14
72	·3090	3074	3057	3040	3024	3007	2990	2974	2957	2940	3	6	8	11	14
73	·2924	2907	2890	2874	2857	2840	2823	2807	2790	2773	3	6	8	11	14
74	·2756	2740	2723	2706	2689	2672	2656	2639	2622	2605	3	6	8	11	14
75	·2588	2571	2554	2538	2521	2504	2487	2470	2453	2436	3	6	8	11	14
76	·2419	2402	2385	2368	2351	2334	2317	2300	2284	2267	3	6	8	11	14
77	·2250	2233	2215	2198	2181	2164	2147	2130	2113	2096	3	6	9	11	14
78	·2079	2062	2045	2028	2011	1994	1977	1959	1942	1925	3	6	9	11	14
79	·1908	1891	1874	1857	1840	1822	1805	1788	1771	1754	3	6	9	11	14
80	·1736	1719	1702	1685	1668	1650	1633	1616	1599	1582	3	6	9	12	14
81	·1564	1547	1530	1513	1495	1478	1461	1444	1426	1409	3	6	9	12	14
82	·1392	1374	1357	1340	1323	1305	1288	1271	1253	1236	3	6	9	12	14
83	·1219	1201	1184	1167	1149	1132	1115	1097	1080	1063	3	6	9	12	14
84	·1045	1028	1011	0993	0976	0958	0941	0924	0906	0889	3	6	9	12	14
85	·0872	0854	0837	0819	0802	0785	0767	0750	0732	0715	3	6	9	12	14
86	·0698	0680	0663	0645	0628	0610	0593	0576	0558	0541	3	6	9	12	15
87	·0523	0506	0488	0471	0454	0436	0419	0401	0384	0366	3	6	9	12	15
88	·0349	0332	0314	0297	0279	0262	0244	0227	0209	0192	3	6	9	12	15
89	·0175	0157	0140	0122	0105	0087	0070	0052	0035	0017	3	6	9	12	15

NATURAL TANGENTS

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
0°	·0000	0017	0035	0052	0070	0087	0105	0122	0140	0157	3	6	9	12	15
1	·0175	0192	0209	0227	0244	0262	0279	0297	0314	0332	3	6	9	12	15
2	·0349	0367	0384	0402	0419	0437	0454	0472	0489	0507	3	6	9	12	15
3	·0524	0542	0559	0577	0594	0612	0629	0647	0664	0682	3	6	9	12	15
4	·0699	0717	0734	0752	0769	0787	0805	0822	0840	0857	3	6	9	12	15
5	·0875	0892	0910	0928	0945	0963	0981	0998	1016	1033	3	6	9	12	15
6	·1051	1069	1086	1104	1122	1139	1157	1175	1192	1210	3	6	9	12	15
7	·1228	1246	1263	1281	1299	1317	1334	1352	1370	1388	3	6	9	12	15
8	·1405	1423	1441	1459	1477	1495	1512	1530	1548	1566	3	6	9	12	15
9	·1584	1602	1620	1638	1655	1673	1691	1709	1727	1745	3	6	9	12	15
10	·1763	1781	1799	1817	1835	1853	1871	1890	1908	1926	3	6	9	12	15
11	·1944	1962	1980	1998	2016	2035	2053	2071	2089	2107	3	6	9	12	15
12	·2126	2144	2162	2180	2199	2217	2235	2254	2272	2290	3	6	9	12	15
13	·2309	2327	2345	2364	2382	2401	2419	2438	2456	2475	3	6	9	12	15
14	·2493	2512	2530	2549	2568	2586	2605	2623	2642	2661	3	6	9	12	16
15	·2679	2698	2717	2736	2754	2773	2792	2811	2830	2849	3	6	9	13	16
16	·2867	2886	2905	2924	2943	2962	2981	3000	3019	3038	3	6	9	13	16
17	·3057	3076	3096	3115	3134	3153	3172	3191	3211	3230	3	6	10	13	16
18	·3249	3269	3288	3307	3327	3346	3365	3385	3404	3424	3	6	10	13	16
19	·3443	3463	3482	3502	3522	3541	3561	3581	3600	3620	3	7	10	13	16
20	·3640	3659	3679	3699	3719	3739	3759	3779	3799	3819	3	7	10	13	17
21	·3839	3859	3879	3899	3919	3939	3959	3979	4000	4020	3	7	10	13	17
22	·4040	4061	4081	4101	4122	4142	4163	4183	4204	4224	3	7	10	14	17
23	·4245	4265	4286	4307	4327	4348	4369	4390	4411	4431	3	7	10	14	17
24	·4452	4473	4494	4515	4536	4557	4578	4599	4621	4642	4	7	11	14	18
25	·4663	4684	4706	4727	4748	4770	4791	4813	4834	4856	4	7	11	14	18
26	·4877	4899	4921	4942	4964	4986	5008	5029	5051	5073	4	7	11	15	18
27	·5095	5117	5139	5161	5184	5206	5228	5250	5272	5295	4	7	11	15	18
28	·5317	5340	5362	5384	5407	5430	5452	5475	5498	5520	4	8	11	15	19
29	·5543	5566	5589	5612	5635	5658	5681	5704	5727	5750	4	8	12	15	19
30	·5774	5797	5820	5844	5867	5890	5914	5938	5961	5985	4	8	12	16	20
31	·6009	6032	6056	6080	6104	6128	6152	6176	6200	6224	4	8	12	16	20
32	·6249	6273	6297	6322	6346	6371	6395	6420	6445	6469	4	8	12	16	20
33	·6494	6519	6544	6569	6594	6619	6644	6669	6694	6720	4	8	13	17	21
34	·6745	6771	6796	6822	6847	6873	6899	6924	6950	6976	4	9	13	17	21
35	·7002	7028	7054	7080	7107	7133	7159	7186	7212	7239	4	9	13	18	22
36	·7265	7292	7319	7346	7373	7400	7427	7454	7481	7508	5	9	14	18	23
37	·7536	7563	7590	7618	7646	7673	7701	7729	7757	7785	5	9	14	18	23
38	·7813	7841	7869	7898	7926	7954	7983	8012	8040	8069	5	9	14	19	24
39	·8098	8127	8156	8185	8214	8243	8273	8302	8332	8361	5	10	15	20	24
40	·8391	8421	8451	8481	8511	8541	8571	8601	8632	8662	5	10	15	20	25
41	·8693	8724	8754	8785	8816	8847	8878	8910	8941	8972	5	10	16	21	26
42	·9004	9036	9067	9099	9131	9163	9195	9228	9260	9293	5	11	16	21	27
43	·9325	9358	9391	9424	9457	9490	9523	9556	9590	9623	6	11	17	22	28
44	·9657	9691	9725	9759	9793	9827	9861	9896	9930	9965	6	11	17	23	29

NATURAL TANGENTS

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
45°	1-0000	0035	0070	0105	0141	0176	0212	0247	0283	0319	6	12	18	24	30
46	1-0355	0392	0428	0464	0501	0538	0575	0612	0649	0686	6	12	18	25	31
47	1-0724	0761	0799	0837	0875	0913	0951	0990	1028	1067	6	13	19	25	32
48	1-1106	1145	1184	1224	1263	1303	1343	1383	1423	1463	7	13	20	27	33
49	1-1504	1544	1585	1626	1667	1708	1750	1792	1833	1875	7	14	21	28	34
50	1-1918	1960	2002	2045	2088	2131	2174	2218	2261	2305	7	14	22	29	36
51	1-2340	2393	2437	2482	2527	2572	2617	2662	2708	2753	8	15	23	30	38
52	1-2799	2846	2892	2938	2985	3032	3079	3127	3175	3222	8	16	24	31	39
53	1-3270	3319	3367	3416	3465	3514	3564	3613	3663	3713	8	16	25	33	41
54	1-3764	3814	3865	3916	3968	4019	4071	4124	4176	4229	9	17	26	34	43
55	1-4281	4335	4388	4442	4496	4550	4605	4659	4715	4770	9	18	27	36	45
56	1-4826	4882	4938	4994	5051	5108	5166	5224	5282	5340	10	19	29	38	48
57	1-5399	5458	5517	5577	5637	5697	5757	5818	5880	5941	10	20	30	40	50
58	1-6003	6066	6128	6191	6255	6319	6383	6447	6512	6577	11	21	32	43	53
59	1-6643	6709	6775	6842	6909	6977	7045	7113	7182	7251	11	23	34	45	56
60	1-7321	7391	7461	7532	7603	7675	7747	7820	7893	7966	12	24	36	48	60
61	1-8040	8115	8190	8265	8341	8418	8495	8572	8650	8728	13	26	38	51	64
62	1-8807	8887	8967	9047	9128	9210	9292	9375	9458	9542	14	27	41	55	68
63	1-9626	9711	9797	9883	9970	0057	0145	0233	0323	0413	15	29	44	58	73
64	2-0503	0594	0686	0778	0872	0965	1060	1155	1251	1348	16	31	47	63	78
65	2-1445	1543	1642	1742	1842	1943	2045	2148	2251	2355	17	34	51	68	85
66	2-2460	2566	2673	2781	2889	2998	3109	3220	3332	3445	18	37	55	73	92
67	2-3559	3673	3789	3906	4023	4142	4262	4383	4504	4627	20	40	60	79	99
68	2-4751	4876	5002	5129	5257	5386	5517	5649	5782	5916	22	43	65	87	108
69	2-6051	6187	6325	6464	6605	6746	6889	7034	7179	7326	24	47	71	95	119
70	2-7475	7625	7776	7929	8083	8239	8397	8556	8716	8878	26	52	78	104	131
71	2-9042	9208	9375	9544	9714	9887	0061	0237	0415	0595	29	58	87	116	145
72	3-0777	0961	1146	1334	1524	1716	1910	2106	2305	2506	32	64	96	129	161
73	3-2709	2914	3122	3332	3544	3759	3977	4197	4420	4646	36	72	108	144	180
74	3-4874	5105	5339	5576	5816	6059	6305	6554	6806	7062	41	81	122	163	204
75	3-7321	7583	7848	8118	8391	8667	8947	9232	9520	9812	46	93	139	186	232
76	4-0108	0408	0713	1022	1335	1653	1976	2303	2635	2972					
77	4-3315	3662	4015	4374	4737	5107	5483	5864	6252	6646					
78	4-7046	7453	7867	8288	8716	9152	9594	0045	0504	0970					
79	5-1446	1929	2422	2924	3435	3955	4486	5026	5578	6140					
80	5-6713	7297	7894	8502	9124	9758	0405	1066	1742	2432					
81	6-3138	3859	4596	5350	6122	6912	7720	8548	9395	0264					
82	7-1154	2066	3002	3962	4947	5958	6996	8062	9158	0285					
83	8-1443	2636	3863	5126	6427	7769	9152	0579	2052	3572					
84	9-514	9-677	9-845	10-02	10-20	10-39	10-58	10-78	10-99	11-20					
85	11-43	11-66	11-91	12-16	12-43	12-71	13-00	13-30	13-62	13-95					
86	14-30	14-67	15-06	15-46	15-89	16-35	16-83	17-34	17-89	18-46					
87	19-08	19-74	20-45	21-20	22-02	22-90	23-86	24-90	26-03	27-27					
88	28-64	30-14	31-82	33-69	35-80	38-19	40-92	44-07	47-74	52-08					
89	57-29	63-66	71-62	81-35	95-49	114-6	143-2	191-0	286-5	573-0					

LOGARITHMS OF SINES

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
0°	-∞	3.2419	5429	7190	8439	9408	0200	0870	1450	1961					
1	2.2419	2832	3210	3558	3880	4179	4459	4723	4971	5206					
2	2.5428	5640	5842	6035	6220	6397	6567	6731	6889	7041					
3	2.7188	7330	7463	7602	7731	7857	7979	8098	8213	8326					
4	2.8436	8543	8647	8749	8849	8946	9042	9135	9226	9315	16	32	48	64	80
5	2.9403	9489	9573	9655	9736	9816	9894	9970	0046	0120	13	26	39	52	65
6	1.0192	0264	0334	0403	0472	0539	0605	0670	0734	0797	11	22	33	44	55
7	1.0859	0920	0981	1040	1099	1157	1214	1271	1326	1381	10	19	29	38	48
8	1.1436	1489	1542	1594	1646	1697	1747	1797	1847	1895	8	17	25	34	42
9	1.1943	1991	2038	2085	2131	2176	2221	2266	2310	2353	8	15	23	30	38
10	1.2397	2439	2482	2524	2565	2606	2647	2687	2727	2767	7	14	20	27	34
11	1.2806	2845	2883	2921	2959	2997	3034	3070	3107	3143	6	12	19	25	31
12	1.3179	3214	3250	3284	3319	3353	3387	3421	3455	3488	6	11	17	23	28
13	1.3521	3554	3586	3618	3650	3682	3713	3745	3775	3806	5	11	16	21	26
14	1.3837	3867	3897	3927	3957	3986	4015	4044	4073	4102	5	10	15	20	24
15	1.4130	4158	4186	4214	4242	4269	4296	4323	4350	4377	5	9	14	18	23
16	1.4403	4430	4456	4482	4508	4533	4559	4584	4609	4634	4	9	13	17	21
17	1.4659	4684	4709	4733	4757	4781	4805	4829	4853	4876	4	8	12	16	20
18	1.4900	4923	4946	4969	4992	5015	5037	5060	5082	5104	4	8	11	15	19
19	1.5126	5148	5170	5192	5213	5235	5256	5278	5299	5320	4	7	11	14	18
20	1.5341	5361	5382	5402	5423	5443	5463	5484	5504	5523	3	7	10	14	17
21	1.5543	5563	5583	5602	5621	5641	5660	5679	5698	5717	3	6	10	13	16
22	1.5736	5754	5773	5792	5810	5828	5847	5865	5883	5901	3	6	9	12	15
23	1.5919	5937	5954	5972	5990	6007	6024	6042	6059	6076	3	6	9	12	15
24	1.6093	6110	6127	6144	6161	6177	6194	6210	6227	6243	3	6	8	11	14
25	1.6259	6276	6292	6308	6324	6340	6356	6371	6387	6403	3	5	8	11	13
26	1.6418	6434	6449	6465	6480	6495	6510	6526	6541	6556	3	5	8	10	13
27	1.6570	6585	6600	6615	6629	6644	6659	6673	6687	6702	2	5	7	10	12
28	1.6716	6730	6744	6759	6773	6787	6801	6814	6828	6842	2	5	7	9	12
29	1.6856	6869	6883	6896	6910	6923	6937	6950	6963	6977	2	4	7	9	11
30	1.6990	7003	7016	7029	7042	7055	7068	7080	7093	7106	2	4	6	9	11
31	1.7118	7131	7144	7156	7168	7181	7193	7205	7218	7230	2	4	6	8	10
32	1.7242	7254	7266	7278	7290	7302	7314	7326	7338	7349	2	4	6	8	10
33	1.7361	7373	7384	7396	7407	7419	7430	7442	7453	7464	2	4	6	8	10
34	1.7476	7487	7498	7509	7520	7531	7542	7553	7564	7575	2	4	6	7	9
35	1.7586	7597	7607	7618	7629	7640	7650	7661	7671	7682	2	4	5	7	9
36	1.7692	7703	7713	7723	7734	7744	7754	7764	7774	7785	2	3	5	7	9
37	1.7795	7805	7815	7825	7835	7844	7854	7864	7874	7884	2	3	5	7	8
38	1.7893	7903	7913	7922	7932	7941	7951	7960	7970	7979	2	3	5	6	8
39	1.7989	7998	8007	8017	8026	8035	8044	8053	8063	8072	2	3	5	6	8
40	1.8081	8090	8099	8108	8117	8125	8134	8143	8152	8161	1	3	4	6	7
41	1.8169	8178	8187	8195	8204	8213	8221	8230	8238	8247	1	3	4	6	7
42	1.8255	8264	8272	8280	8289	8297	8305	8313	8322	8330	1	3	4	6	7
43	1.8338	8346	8354	8362	8370	8378	8386	8394	8402	8410	1	3	4	5	7
44	1.8418	8426	8433	8441	8449	8457	8464	8472	8480	8487	1	3	4	5	6

LOGARITHMS OF SINES

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
45°	̄.8495	8502	8510	8517	8525	8532	8540	8547	8555	8562	1	2	4	5	6
46	̄.8569	8577	8584	8591	8598	8606	8613	8620	8627	8634	1	2	4	5	6
47	̄.8641	8648	8655	8662	8669	8676	8683	8690	8697	8704	1	2	3	5	6
48	̄.8711	8718	8724	8731	8738	8745	8751	8758	8765	8771	1	2	3	4	6
49	̄.8778	8784	8791	8797	8804	8810	8817	8823	8830	8836	1	2	3	4	5
50	̄.8843	8849	8855	8862	8868	8874	8880	8887	8893	8899	1	2	3	4	5
51	̄.8905	8911	8917	8923	8929	8935	8941	8947	8953	8959	1	2	3	4	5
52	̄.8965	8971	8977	8983	8989	8995	9000	9006	9012	9018	1	2	3	4	5
53	̄.9023	9029	9035	9041	9046	9052	9057	9063	9069	9074	1	2	3	4	5
54	̄.9080	9085	9091	9096	9101	9107	9112	9118	9123	9128	1	2	3	4	5
55	̄.9134	9139	9144	9149	9155	9160	9165	9170	9175	9181	1	2	3	3	4
56	̄.9186	9191	9196	9201	9206	9211	9216	9221	9226	9231	1	2	3	3	4
57	̄.9236	9241	9246	9251	9255	9260	9265	9270	9275	9279	1	2	2	3	4
58	̄.9284	9289	9294	9298	9303	9308	9312	9317	9322	9326	1	2	2	3	4
59	̄.9331	9335	9340	9344	9349	9353	9358	9362	9367	9371	1	1	2	3	4
60	̄.9375	9380	9384	9388	9393	9397	9401	9406	9410	9414	1	1	2	3	4
61	̄.9418	9422	9427	9431	9435	9439	9443	9447	9451	9455	1	1	2	3	3
62	̄.9459	9463	9467	9471	9475	9479	9483	9487	9491	9495	1	1	2	3	3
63	̄.9499	9503	9506	9510	9514	9518	9522	9525	9529	9533	1	1	2	3	3
64	̄.9537	9540	9544	9548	9551	9555	9558	9562	9566	9569	1	1	2	2	3
65	̄.9573	9576	9580	9583	9587	9590	9594	9597	9601	9604	1	1	2	2	3
66	̄.9607	9611	9614	9617	9621	9624	9627	9631	9634	9637	1	1	2	2	3
67	̄.9640	9643	9647	9650	9653	9656	9659	9662	9666	9669	1	1	2	2	3
68	̄.9672	9675	9678	9681	9684	9687	9690	9693	9696	9699	0	1	1	2	2
69	̄.9702	9704	9707	9710	9713	9716	9719	9722	9724	9727	0	1	1	2	2
70	̄.9730	9733	9735	9738	9741	9743	9746	9749	9751	9754	0	1	1	2	2
71	̄.9757	9759	9762	9764	9767	9770	9772	9775	9777	9780	0	1	1	2	2
72	̄.9782	9785	9787	9789	9792	9794	9797	9799	9801	9804	0	1	1	2	2
73	̄.9806	9808	9811	9813	9815	9817	9820	9822	9824	9826	0	1	1	1	2
74	̄.9828	9831	9833	9835	9837	9839	9841	9843	9845	9847	0	1	1	1	2
75	̄.9849	9851	9853	9855	9857	9859	9861	9863	9865	9867	0	1	1	1	2
76	̄.9869	9871	9873	9875	9876	9878	9880	9882	9884	9885	0	1	1	1	2
77	̄.9887	9889	9891	9892	9894	9896	9897	9899	9901	9902	0	1	1	1	1
78	̄.9904	9906	9907	9909	9910	9912	9913	9915	9916	9918	0	1	1	1	1
79	̄.9919	9921	9922	9924	9925	9927	9928	9929	9931	9932	0	0	1	1	1
80	̄.9934	9935	9936	9937	9939	9940	9941	9943	9944	9945	0	0	1	1	1
81	̄.9946	9947	9949	9950	9951	9952	9953	9954	9955	9956	0	0	1	1	1
82	̄.9958	9959	9960	9961	9962	9963	9964	9965	9966	9967	0	0	1	1	1
83	̄.9968	9968	9969	9970	9971	9972	9973	9974	9975	9975	0	0	0	1	1
84	̄.9976	9977	9978	9978	9979	9980	9981	9981	9982	9983	0	0	0	0	1
85	̄.9983	9984	9985	9985	9986	9987	9987	9988	9988	9989	0	0	0	0	0
86	̄.9989	9990	9990	9991	9991	9992	9992	9993	9993	9994	0	0	0	0	0
87	̄.9994	9994	9995	9995	9996	9996	9996	9996	9997	9997	0	0	0	0	0
88	̄.9997	9998	9998	9998	9998	9999	9999	9999	9999	9999	0	0	0	0	0
89	̄.9999	9999	0000	0000	0000	0000	0000	0000	0000	0000	0	0	0	0	0

LOGARITHMS OF COSINES

Subtract differences

											Differences				
	O'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
0°	0.0000	0000	0000	0000	0000	0000	0000	0000	0000	9999	0	0	0	0	0
1	1.9999	9999	9999	9999	9999	9999	9998	9998	9998	9998	0	0	0	0	0
2	1.9997	9997	9997	9996	9996	9996	9996	9995	9995	9994	0	0	0	0	0
3	1.9994	9994	9993	9993	9992	9992	9991	9991	9990	9990	0	0	0	0	0
4	1.9989	9989	9988	9988	9987	9987	9986	9985	9985	9984	0	0	0	0	0
5	1.9983	9983	9982	9981	9981	9980	9979	9978	9978	9977	0	0	0	0	1
6	1.9976	9975	9975	9974	9973	9972	9971	9970	9969	9968	0	0	0	1	1
7	1.9968	9967	9966	9965	9964	9963	9962	9961	9960	9959	0	0	1	1	1
8	1.9958	9956	9955	9954	9953	9952	9951	9950	9949	9947	0	0	1	1	1
9	1.9946	9945	9944	9943	9941	9940	9939	9937	9936	9935	0	0	1	1	1
10	1.9934	9932	9931	9929	9928	9927	9925	9924	9922	9921	0	0	1	1	1
11	1.9919	9918	9916	9915	9913	9912	9910	9909	9907	9906	0	1	1	1	1
12	1.9904	9902	9901	9899	9897	9896	9894	9892	9891	9889	0	1	1	1	1
13	1.9887	9885	9884	9882	9880	9878	9876	9875	9873	9871	0	1	1	1	2
14	1.9869	9867	9865	9863	9861	9859	9857	9855	9853	9851	0	1	1	1	2
15	1.9849	9847	9845	9843	9841	9839	9837	9835	9833	9831	0	1	1	1	2
16	1.9828	9826	9824	9822	9820	9817	9815	9813	9811	9808	0	1	1	1	2
17	1.9806	9804	9801	9799	9797	9794	9792	9789	9787	9785	0	1	1	2	2
18	1.9782	9780	9777	9775	9772	9770	9767	9764	9762	9759	0	1	1	2	2
19	1.9757	9754	9751	9749	9746	9743	9741	9738	9735	9733	0	1	1	2	2
20	1.9730	9727	9724	9722	9719	9716	9713	9710	9707	9704	0	1	1	2	2
21	1.9702	9699	9696	9693	9690	9687	9684	9681	9678	9675	0	1	1	2	2
22	1.9672	9669	9666	9662	9659	9656	9653	9650	9647	9643	1	1	2	2	3
23	1.9640	9637	9634	9631	9627	9624	9621	9617	9614	9611	1	1	2	2	3
24	1.9607	9604	9601	9597	9594	9590	9587	9583	9580	9576	1	1	2	2	3
25	1.9573	9569	9566	9562	9558	9555	9551	9548	9544	9540	1	1	2	2	3
26	1.9537	9533	9529	9525	9522	9518	9514	9510	9506	9503	1	1	2	3	3
27	1.9499	9495	9491	9487	9483	9479	9475	9471	9467	9463	1	1	2	3	3
28	1.9459	9455	9451	9447	9443	9439	9435	9431	9427	9422	1	1	2	3	3
29	1.9418	9414	9410	9406	9401	9397	9393	9388	9384	9380	1	1	2	3	4
30	1.9375	9371	9367	9362	9358	9353	9349	9344	9340	9335	1	1	2	3	4
31	1.9331	9326	9322	9317	9312	9308	9303	9298	9294	9289	1	2	2	3	4
32	1.9284	9279	9275	9270	9265	9260	9255	9251	9246	9241	1	2	2	3	4
33	1.9236	9231	9226	9221	9216	9211	9206	9201	9196	9191	1	2	3	3	4
34	1.9186	9181	9175	9170	9165	9160	9155	9149	9144	9139	1	2	3	3	4
35	1.9134	9128	9123	9118	9112	9107	9101	9096	9091	9085	1	2	3	4	5
36	1.9080	9074	9069	9063	9057	9052	9046	9041	9035	9029	1	2	3	4	5
37	1.9023	9018	9012	9006	9000	8995	8989	8983	8977	8971	1	2	3	4	5
38	1.8965	8959	8953	8947	8941	8935	8929	8923	8917	8911	1	2	3	4	5
39	1.8905	8899	8893	8887	8880	8874	8868	8862	8855	8849	1	2	3	4	5
40	1.8843	8836	8830	8823	8817	8810	8804	8797	8791	8784	1	2	3	4	5
41	1.8778	8771	8765	8758	8751	8745	8738	8731	8724	8718	1	2	3	5	6
42	1.8711	8704	8697	8690	8683	8676	8669	8662	8655	8648	1	2	3	5	6
43	1.8641	8634	8627	8620	8613	8606	8598	8591	8584	8577	1	2	4	5	6
44	1.8569	8562	8555	8547	8540	8532	8525	8517	8510	8502	1	2	4	5	6

LOGARITHMS OF COSINES

Subtract differences

										Differences				
	O'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4' 5'
45°	1-8495	8487	8480	8472	8464	8457	8449	8441	8433	8426	1	3	4	5 6
46	1-8418	8410	8402	8394	8386	8378	8370	8362	8354	8346	1	3	4	5 7
47	1-8338	8330	8322	8313	8305	8297	8289	8280	8272	8264	1	3	4	6 7
48	1-8255	8247	8238	8230	8221	8213	8204	8195	8187	8178	1	3	4	6 7
49	1-8169	8161	8152	8143	8134	8125	8117	8108	8099	8090	1	3	4	6 7
50	1-8081	8072	8063	8053	8044	8035	8026	8017	8007	7998	2	3	5	6 8
51	1-7989	7979	7970	7960	7951	7941	7932	7922	7913	7903	2	3	5	6 8
52	1-7893	7884	7874	7864	7854	7844	7835	7825	7815	7805	2	3	5	7 8
53	1-7795	7785	7774	7764	7754	7744	7734	7723	7713	7703	2	3	5	7 9
54	1-7692	7682	7671	7661	7650	7640	7629	7618	7607	7597	2	4	5	7 9
55	1-7586	7575	7564	7553	7542	7531	7520	7509	7498	7487	2	4	6	7 9
56	1-7476	7464	7453	7442	7430	7419	7407	7396	7384	7373	2	4	6	8 10
57	1-7361	7349	7338	7326	7314	7302	7290	7278	7266	7254	2	4	6	8 10
58	1-7242	7230	7218	7205	7193	7181	7168	7156	7144	7131	2	4	6	8 10
59	1-7118	7106	7093	7080	7068	7055	7042	7029	7016	7003	2	4	6	9 11
60	1-6990	6977	6963	6950	6937	6923	6910	6896	6883	6869	2	4	7	9 11
61	1-6856	6842	6828	6814	6801	6787	6773	6759	6744	6730	2	5	7	9 12
62	1-6716	6702	6687	6673	6659	6644	6629	6615	6600	6585	2	5	7	10 12
63	1-6570	6556	6541	6526	6510	6495	6480	6465	6449	6434	2	5	8	10 13
64	1-6418	6403	6387	6371	6356	6340	6324	6308	6292	6276	3	5	8	11 13
65	1-6259	6243	6227	6210	6194	6177	6161	6144	6127	6110	3	6	8	11 14
66	1-6093	6076	6059	6042	6024	6007	5990	5972	5954	5937	3	6	9	12 15
67	1-5919	5901	5883	5865	5847	5828	5810	5792	5773	5754	3	6	9	12 15
68	1-5736	5717	5698	5679	5660	5641	5621	5602	5583	5563	3	6	10	13 16
69	1-5543	5523	5504	5484	5463	5443	5423	5402	5382	5361	3	7	10	14 17
70	1-5341	5320	5299	5278	5256	5235	5213	5192	5170	5148	4	7	11	14 18
71	1-5126	5104	5082	5060	5037	5015	4992	4969	4946	4923	4	8	11	15 19
72	1-4900	4876	4853	4829	4805	4781	4757	4733	4709	4684	4	8	12	16 20
73	1-4659	4634	4609	4584	4559	4533	4508	4482	4456	4430	4	9	13	17 21
74	1-4403	4377	4350	4323	4296	4269	4242	4214	4186	4158	5	9	14	18 23
75	1-4130	4102	4073	4044	4015	3986	3957	3927	3897	3867	5	10	15	20 24
76	1-3837	3806	3775	3745	3713	3682	3650	3618	3586	3554	5	11	16	21 26
77	1-3521	3488	3455	3421	3387	3353	3319	3284	3250	3214	6	11	17	23 28
78	1-3179	3143	3107	3070	3034	2997	2959	2921	2883	2845	6	12	19	25 31
79	1-2806	2767	2727	2687	2647	2606	2565	2524	2482	2439	7	14	20	27 34
80	1-2397	2353	2310	2266	2221	2176	2131	2085	2038	1991	8	15	23	30 38
81	1-1943	1895	1847	1797	1747	1697	1646	1594	1542	1489	8	17	25	34 42
82	1-1436	1381	1326	1271	1214	1157	1099	1040	981	920	10	19	29	38 48
83	1-0859	0797	0734	0670	0605	0539	0472	0403	0334	0264	11	22	33	44 55
84	1-0192	0120	0046	9970	9894	9816	9736	9655	9573	9489	13	26	39	52 65
85	2-9403	9315	9226	9135	9042	8946	8849	8749	8647	8543	16	32	48	61 80
86	2-8436	8326	8213	8098	7979	7857	7731	7602	7468	7330				
87	2-7188	7041	6889	6731	6567	6397	6220	6035	5842	5640				
88	2-5428	5206	4971	4723	4459	4179	3880	3558	3210	2832				
89	2-2419	1961	1450	0870	0200	9408	8439	7190	5429	2419				

LOGARITHMS OF TANGENTS

											Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'	1'	2'	3'	4'	5'
0°	—∞	3-2419	5429	7190	8439	9409	0200	0870	1450	1962					
1	2-2419	2833	3211	3559	3881	4181	4461	4725	4973	5208					
2	2-5431	5643	5845	6038	6223	6401	6571	6736	6894	7046					
3	2-7194	7337	7475	7609	7739	7865	7988	8107	8223	8336					
4	2-8446	8554	8659	8762	8862	8960	9056	9150	9241	9331	16	32	48	64	81
5	2-9420	9506	9591	9674	9756	9836	9915	9992	0068	0143	13	26	40	53	66
6	1-0216	0289	0360	0430	0499	0567	0633	0699	0764	0828	11	22	34	45	56
7	1-0891	0954	1015	1076	1135	1194	1252	1310	1367	1423	10	20	29	39	49
8	1-1478	1533	1587	1640	1693	1745	1797	1848	1898	1948	9	17	26	35	43
9	1-1997	2046	2094	2142	2189	2236	2282	2328	2374	2419	8	16	23	31	39
10	1-2463	2507	2551	2594	2637	2680	2722	2764	2805	2846	7	14	21	28	35
11	1-2887	2927	2967	3006	3046	3085	3123	3162	3200	3237	6	13	19	26	32
12	1-3275	3312	3349	3385	3422	3458	3493	3529	3564	3599	6	12	18	24	30
13	1-3634	3668	3702	3736	3770	3804	3837	3870	3903	3935	6	11	17	22	28
14	1-3968	4000	4032	4064	4095	4127	4158	4189	4220	4250	5	10	16	21	26
15	1-4281	4311	4341	4371	4400	4430	4459	4488	4517	4546	5	10	15	20	25
16	1-4575	4603	4632	4660	4688	4716	4744	4771	4799	4826	5	9	14	19	23
17	1-4853	4880	4907	4934	4961	4987	5014	5040	5066	5092	4	9	13	18	22
18	1-5118	5143	5169	5195	5220	5245	5270	5295	5320	5345	4	8	13	17	21
19	1-5370	5394	5419	5443	5467	5491	5516	5539	5563	5587	4	8	12	16	20
20	1-5611	5634	5658	5681	5704	5727	5750	5773	5796	5819	4	8	12	15	19
21	1-5842	5864	5887	5909	5932	5954	5976	5998	6020	6042	4	7	11	15	19
22	1-6064	6086	6108	6129	6151	6172	6194	6215	6236	6257	4	7	11	14	18
23	1-6279	6300	6321	6341	6362	6383	6404	6424	6445	6465	3	7	10	14	17
24	1-6486	6506	6527	6547	6567	6587	6607	6627	6647	6667	3	7	10	13	17
25	1-6687	6706	6726	6746	6765	6785	6804	6824	6843	6863	3	7	10	13	16
26	1-6882	6901	6920	6939	6958	6977	6996	7015	7034	7053	3	6	9	13	16
27	1-7072	7090	7109	7128	7146	7165	7183	7202	7220	7238	3	6	9	12	15
28	1-7257	7275	7293	7311	7330	7348	7366	7384	7402	7420	3	6	9	12	15
29	1-7438	7455	7473	7491	7509	7526	7544	7562	7579	7597	3	6	9	12	15
30	1-7614	7632	7649	7667	7684	7701	7719	7736	7753	7771	3	6	9	12	14
31	1-7788	7805	7822	7839	7856	7873	7890	7907	7924	7941	3	6	9	11	14
32	1-7958	7975	7992	8008	8025	8042	8059	8075	8092	8109	3	6	8	11	14
33	1-8125	8142	8158	8175	8191	8208	8224	8241	8257	8274	3	5	8	11	14
34	1-8290	8306	8323	8339	8355	8371	8388	8404	8420	8436	3	5	8	11	14
35	1-8452	8468	8484	8501	8517	8533	8549	8565	8581	8597	3	5	8	11	13
36	1-8613	8629	8644	8660	8676	8692	8708	8724	8740	8755	3	5	8	11	13
37	1-8771	8787	8803	8818	8834	8850	8865	8881	8897	8912	3	5	8	10	13
38	1-8928	8944	8959	8975	8990	9006	9022	9037	9053	9068	3	5	8	10	13
39	1-9084	9099	9115	9130	9146	9161	9176	9192	9207	9223	3	5	8	10	13
40	1-9238	9254	9269	9284	9300	9315	9330	9346	9361	9376	3	5	8	10	13
41	1-9392	9407	9422	9438	9453	9468	9483	9499	9514	9529	3	5	8	10	13
42	1-9544	9560	9575	9590	9605	9621	9636	9651	9666	9681	3	5	8	10	13
43	1-9697	9712	9727	9742	9757	9773	9788	9803	9818	9833	3	5	8	10	13
44	1-9848	9864	9879	9894	9909	9924	9939	9955	9970	9985	3	5	8	10	13

LOGARITHMS OF TANGENTS

													Differences				
	0'	6'	12'	18'	24'	30'	36'	42'	48'	54'			1'	2'	3'	4'	5'
45°	·0000	0015	0030	0045	0061	0076	0091	0106	0121	0136			3	5	8	10	13
46	·0152	0167	0182	0197	0212	0228	0243	0258	0273	0288			3	5	8	10	13
47	·0303	0319	0334	0349	0364	0379	0395	0410	0425	0440			3	5	8	10	13
48	·0456	0471	0486	0501	0517	0532	0547	0562	0578	0593			3	5	8	10	13
49	·0603	0624	0639	0654	0670	0685	0700	0716	0731	0746			3	5	8	10	13
50	·0762	0777	0793	0808	0824	0839	0854	0870	0885	0901			3	5	8	10	13
51	·0916	0932	0947	0963	0978	0994	1010	1025	1041	1056			3	5	8	10	13
52	·1072	1088	1103	1119	1135	1150	1166	1182	1197	1213			3	5	8	10	13
53	·1229	1245	1260	1276	1292	1308	1324	1340	1356	1371			3	5	8	11	13
54	·1387	1403	1419	1435	1451	1467	1483	1499	1516	1532			3	5	8	11	13
55	·1548	1564	1580	1596	1612	1629	1645	1661	1677	1694			3	5	8	11	14
56	·1710	1726	1743	1759	1776	1792	1809	1825	1842	1858			3	5	8	11	14
57	·1875	1891	1908	1925	1941	1958	1975	1992	2008	2025			3	6	8	11	14
58	·2042	2059	2076	2093	2110	2127	2144	2161	2178	2195			3	6	9	11	14
59	·2212	2229	2247	2264	2281	2299	2316	2333	2351	2368			3	6	9	12	14
60	·2386	2403	2421	2438	2456	2474	2491	2509	2527	2545			3	6	9	12	15
61	·2562	2580	2598	2616	2634	2652	2670	2689	2707	2725			3	6	9	12	15
62	·2743	2762	2780	2798	2817	2835	2854	2872	2891	2910			3	6	9	12	15
63	·2928	2947	2966	2985	3004	3023	3042	3061	3080	3099			3	6	9	13	16
64	·3118	3137	3157	3176	3196	3215	3235	3254	3274	3294			3	7	10	13	16
65	·3313	3333	3353	3373	3393	3413	3433	3453	3473	3494			3	7	10	13	17
66	·3514	3535	3555	3576	3596	3617	3638	3659	3679	3700			3	7	10	14	17
67	·3721	3743	3764	3785	3806	3828	3849	3871	3892	3914			4	7	11	14	18
68	·3936	3958	3980	4002	4024	4046	4068	4091	4113	4136			4	7	11	15	19
69	·4158	4181	4204	4227	4250	4273	4296	4319	4342	4366			4	8	12	15	19
70	·4389	4413	4437	4461	4484	4509	4533	4557	4581	4606			4	8	12	16	20
71	·4630	4655	4680	4705	4730	4755	4780	4805	4831	4857			4	8	13	17	21
72	·4882	4908	4934	4960	4986	5013	5039	5066	5093	5120			4	9	13	18	22
73	·5147	5174	5201	5229	5256	5284	5312	5340	5368	5397			5	9	14	19	23
74	·5425	5454	5483	5512	5541	5570	5600	5629	5659	5689			5	10	15	20	25
75	·5719	5750	5780	5811	5842	5873	5905	5936	5968	6000			5	10	16	21	26
76	·6032	6065	6097	6130	6163	6196	6230	6264	6298	6332			6	11	17	22	28
77	·6366	6401	6436	6471	6507	6542	6578	6615	6651	6688			6	12	18	24	30
78	·6725	6763	6800	6838	6877	6915	6954	6994	7033	7073			6	13	19	26	32
79	·7113	7154	7195	7236	7278	7320	7363	7406	7449	7493			7	14	21	28	35
80	·7537	7581	7626	7672	7718	7764	7811	7858	7906	7954			8	16	23	31	39
81	·8003	8052	8102	8152	8203	8255	8307	8360	8413	8467			9	17	26	35	43
82	·8522	8577	8633	8690	8718	8806	8865	8924	8985	9046			10	20	29	39	49
83	·9105	9172	9236	9301	9367	9433	9501	9570	9640	9711			11	22	34	45	56
84	·9784	9857	9932	0008	0085	0164	0244	0326	0409	0494			13	26	40	53	66
85	1-0580	0669	0759	0850	0944	1040	1138	1238	1341	1446			16	32	48	64	81
86	1-1554	1664	1777	1893	2012	2135	2261	2391	2526	2663							
87	1-2806	2954	3106	3264	3429	3599	3777	3962	4155	4357							
88	1-4569	4792	5027	5275	5539	5819	6119	6441	6789	7167							
89	1-7581	8038	8550	9130	9800	0591	1561	2810	4571	7581							

ANSWERS TO EXERCISES

ANSWERS TO EXERCISES

Page 10

- | | |
|------------------------------|------------------|
| 1. (a) $48^\circ 52' 40''$. | 2. (a) 0.885917. |
| (b) $289^\circ 36' 19''$. | (b) 0.391065. |
| (c) $96^\circ 13' 55''$. | (c) 1.584309. |

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|--|--------------------------------------|
| 1. $\frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}, 2\pi$. | 5. 1 : 3, 1 : 2, 1 : 5. |
| 2. $60^\circ, 150^\circ, 135^\circ, 225^\circ, 270^\circ, 300^\circ$. | 6. 1.2171 radians. |
| 3. 30° . | 7. $57^\circ 17' 45''$. |
| 4. (a) $\frac{\pi}{3}, 60^\circ$; (b) $\frac{5\pi}{6}, 150^\circ$. | 8. $121\frac{1}{4}, 98\frac{3}{4}$. |

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($\pi = 3.142$)

- | | |
|-----------------------------|--|
| 1. 38.49 sq. in., 21.99 in. | 12. 28.28 sq. in. (red), 56.56 sq. in. (blue). |
| 2. 4.18 in. | 13. 14.28 sq. in. |
| 3. 4.71 ft. | 14. $\frac{3}{5}\pi, \frac{3}{4}\pi, \frac{4}{5}\pi$. |
| 4. 110.9 sq. cm. | 15. 2164 miles. |
| 5. 15.53 ft. | 16. 1222 miles. |
| 6. $114\frac{7}{12}$. | 17. $25^\circ 12'$. |
| 7. 13.75 in. | 18. $\frac{1}{3}\pi a^2$. |
| 8. 5.46 cm. | 20. $a^2 \left(\frac{2}{3}\pi + \sqrt{3} \right)$. |
| 9. 28.87 cm. | |
| 10. 1 : 8. | |
| 11. 35.35 sq. ft. | |

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- | | | | |
|-----------|-----------|------------------|-------------------|
| 1. 0.54. | 4. -1.18. | 7. 20° . | 10. 227° . |
| 2. -0.59. | 5. 4.44. | 8. 157° . | 11. 297° . |
| 3. -2.14. | 6. -0.47. | 9. 287° . | 12. 219° . |

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- | | | | | |
|---------------------|---------------------------|----------------------|-----------------------|---------------------|
| 1. 1. | 5. $\frac{1}{2}$. | 7. 0. | 11. $\frac{3}{4}$. | 14. -1. |
| 2. $\frac{1}{2}$. | | 8. $-2\frac{1}{2}$. | 12. $-3\frac{1}{2}$. | 15. 0. |
| 3. 0. | 6. $\frac{\sqrt{3}}{2}$. | 9. 1. | 13. 1. | 16. $\frac{1}{2}$. |
| 4. $2\frac{1}{2}$. | | 10. 1. | | |

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1. $\tan A = \pm \frac{3}{4}$, $\operatorname{cosec} A = \pm \frac{5}{3}$.
2. $\sin A = \pm \frac{7}{25}$, $\cos A = \pm \frac{24}{25}$.
3. $\cos A = \mp \frac{40}{41}$, $\cot A = \pm \frac{40}{9}$.
4. $\operatorname{cosec} A = \pm \frac{17}{8}$, $\tan A = \mp \frac{8}{15}$.
5. $\sin A = -\frac{7}{25}$, $\sec A = -\frac{25}{24}$.
6. $\sin A = -\frac{11}{61}$, $\tan A = -\frac{11}{60}$.
7. $\cos A = \frac{84}{85}$, $A = 351^\circ$.
8. $\operatorname{cosec} A = \pm \frac{17}{8}$, 4th quadrant.
9. $\tan A = \pm \frac{12}{5}$, 3rd quadrant.
10. $\sec A = \frac{41}{9}$.
11. $\sin A = \frac{144}{145}$.
12. 260° .
13. -0.72 .
14. 0.29 .
15. 0.62 .
16. -0.96 .

$$17. \sin^2 A = \frac{4t^2}{(1+t^2)^2}, \tan^2 A = \frac{4t^2}{(1-t^2)^2}.$$

$$18. \sin^2 A = \left\{ \frac{2n(n+1)}{2n^2+2n+1} \right\}^2, \sec^2 A = \left\{ \frac{2n^2+2n+1}{2n+1} \right\}^2.$$

$$19. \cos^2 A = \left(\frac{a-b}{a+b} \right)^2, \cot^2 A = \frac{a^2-2ab+b^2}{4ab}.$$

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|--------------------------|-------------------------|------------------------------------|----------------------|
| 1. $\frac{1}{\sqrt{2}}$ | 6. $-\sqrt{2}$ | 11. $-\frac{1}{2}$ | 18. $\sin 10^\circ$ |
| 2. 2. | 7. -2. | 12. $-\sqrt{2}$ | 19. $-\cot 43^\circ$ |
| 3. $\sqrt{3}$ | 8. $\frac{1}{\sqrt{2}}$ | 13. $-\cos 19^\circ$ | 20. $\cos 35^\circ$ |
| 4. -1. | 9. $\frac{1}{\sqrt{3}}$ | 14. $-\tan 31^\circ$ | 21. $-\sec 28^\circ$ |
| 5. $-\frac{\sqrt{3}}{2}$ | 10. $\sqrt{3}$ | 15. $\cot 25^\circ$ | 22. $\sin 10^\circ$ |
| | | 16. $\sec 37^\circ$ | 23. $-\sec 32^\circ$ |
| | | 17. $\operatorname{cosec} 7^\circ$ | 24. $\cot 20^\circ$ |

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1. $A = 30^\circ$, $B = 60^\circ$, $c = 20$.
2. $B = 30^\circ$, $a = 200$, $c = 100\sqrt{3}$.
3. $A = 33^\circ$, $a = 1.74$, $b = 2.68$.
4. $A = 61^\circ 21'$, $C = 28^\circ 39'$, $a = 8.60$.
5. $A = 31^\circ 51'$, $B = 58^\circ 9'$, $c = 112$.
6. $B = 7^\circ$, $a = 861$, $c = 855$.
7. $69^\circ 42'$.
8. 6.07 .
9. 22.1 .
10. $6^\circ 32'$.

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1. 50 ft.
2. 60° .
3. $40\sqrt{3}$ ft.
4. 40 ft.
5. 30° .
6. 2 miles.

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|---|--------------------------------------|-------------------------------|
| 7. $63^\circ 26'$. | 15. 0.41 mile; 1.08 miles. | 24. 107 ft., 113 ft., 108 ft. |
| 8. $19^\circ 59'$. | | |
| 9. 231 ft. | 16. 56.0 ft. | 27. $0.31 : 1$. |
| 10. $1 : 2$. | 18. 34° . | 28. 6.14×10^3 ft. |
| 11. $25^\circ 1'$. | 19. 2.14×10^3 ft. | 29. 161 ft. |
| 12. $58^\circ 20'$. | 20. $40^\circ 54'$ W. of N. | 30. 1.14×10^3 ft. |
| 13. 11.0 ft. | 21. $44^\circ 43'$. | 31. $21^\circ 57'$. |
| 14. $67^\circ 23'$ E. of N. or W. of N. | 22. 26.9 ft. | 32. 1.08 miles. |
| | 23. $68^\circ, 68^\circ, 44^\circ$. | 33. 3.6 miles, 4.3 miles. |

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|----------------------------|-----------------------------|-----------------------------|-----------------------------|
| 1. 45° . | 9. No value. | 16. $22\frac{1}{2}^\circ$. | 24. 1.89 radians. |
| 2. 45° . | 10. 52° . | 17. 39° . | 25. 0.91 radian. |
| 3. 41° . | 11. 54. | 18. $56\frac{1}{2}^\circ$. | 26. $49\frac{1}{2}^\circ$. |
| 4. $13^\circ, 77^\circ$. | 12. 23° . | 19. $22\frac{1}{2}^\circ$. | 27. 34° . |
| 5. $47\frac{1}{2}^\circ$. | 13. $24^\circ, 66^\circ$. | 20. 27° . | 28. $28\frac{1}{2}^\circ$. |
| 6. 284° . | 14. $51\frac{1}{2}^\circ$. | 21. 144° . | 29. 50. |
| 7. 38° . | 15. $74\frac{1}{2}^\circ$. | 22. $-19\frac{1}{2}$. | 30. $47\frac{1}{2}$. |
| 8. 32° . | | | |

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|--|--|
| 1. $\sin 25^\circ + \sin 9^\circ$. | 12. $\frac{1}{2} [\cos (\theta - \phi) - \cos (\theta + \phi)]$. |
| 2. $\sin 25^\circ - \sin 9^\circ$. | 13. $\sin (5A + 5B) - \sin (A - B)$. |
| 3. $\cos 17^\circ - \cos 29^\circ$. | 14. $\cos 2y - \cos 2x$. |
| 4. $\cos 66^\circ + \cos 20^\circ$. | 15. $\sin \theta + \sin \phi$. |
| 5. $\frac{1}{2} (\sin 86^\circ - \sin 36^\circ)$. | 16. $\sin 4A + \sin 2B$. |
| 6. $\frac{1}{2} (\cos 27^\circ - \cos 71^\circ)$. | 17. $\cos (A + B + C) + \cos C$. |
| 7. $\sin 3A + \sin A$. | 18. $\cos z - \cos (x - y + z)$. |
| 8. $\cos 3A + \cos A$. | 19. $\frac{1}{2} \left[\sin \frac{7\pi}{12} + \sin \left(\frac{\pi}{12} + 2\theta \right) \right]$. |
| 9. $\sin 2A - \sin A$. | 20. $\frac{1}{2} \left[\cos \left(\frac{3\pi}{2} - \phi \right) + \cos \left(\frac{\pi}{6} - 5\phi \right) \right]$. |
| 10. $\cos A - \cos 4A$. | |
| 11. $\sin (2A + B) + \sin (2A - B)$. | |

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|--------------------------|--|---|
| 1. $2 \sin 2A \cos A$. | 3. $2 \cos 5A \cos 2A$. | 5. $2 \sin 6\theta \cos \theta$. |
| 2. $2 \sin 6A \sin 2A$. | 4. $2 \cos \frac{7A}{2} \sin \frac{3A}{2}$. | 6. $2 \cos \frac{15\theta}{2} \cos \frac{3\theta}{2}$. |

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|--|--|
| 7. $2 \sin 4A \sin 2A$. | 12. $2 \sin 32^\circ \sin 27^\circ$. |
| 8. $2 \cos 2(B + A) \sin(B - A)$. | 13. $2 \cos 28^\circ \cos 5^\circ$. |
| 9. $2 \sin 3A \sin \frac{A}{2}$. | 14. $2 \cos 59^\circ \sin 5^\circ$. |
| 10. $2 \sin 45^\circ \cos(45^\circ - A)$. | 15. $2 \cos 5^\circ \sin 1^\circ$. |
| 11. $2 \sin 31^\circ \cos 6^\circ$. | 16. $2 \sin 28\frac{1}{2}^\circ \cos 9\frac{1}{2}^\circ$. |

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|----------|----------|----------|---------------------|---------------------|
| 1. 8·97. | 3. 2·89. | 5. 3·01. | 7. 12·1. | 9. $78^\circ 28'$. |
| 2. 10. | 4. 8·72. | 6. 9·26. | 8. $41^\circ 24'$. | |

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|-----------|------------------------------|----------------------|-----------------------|
| 1. 26·1. | 7. 0·306. | 25. (a) 0·707, | 30. 1·18. |
| 2. 7·19. | 8. 26·9. | (b) 3. | 31. $70^\circ 4'$. |
| 3. 9·23. | 9. 0·148. | 26. 10·5. | 32. $61^\circ 16'$ or |
| 4. -9·66. | 10. $2\cdot70 \times 10^3$. | 27. $20^\circ 15'$. | $118^\circ 44'$. |
| 5. 0·400. | 11. 2·34. | 28. 16·2. | 33. $39^\circ 0'$. |
| 6. 0·156. | 12. 12·4. | 29. 342. | 34. $96^\circ 48'$. |
35. $A = 57^\circ 19'$, $B = 37^\circ 58'$, $C = 84^\circ 44'$.

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|---------------------------|------------------------------|------------------------------|
| 1. $C = 66^\circ 30'$, | $a = 31\cdot3$, | $b = 22\cdot0$. |
| 2. $B = 61^\circ 45'$, | $b = 26\cdot0$, | $c = 18\cdot5$. |
| 3. $A = 32^\circ 26'$, | $a = 12\cdot9$, | $b = 23\cdot8$. |
| 4. $B = 25^\circ 18'$, | $a = 12\cdot6$, | $c = 17\cdot8$. |
| 5. $A = 40^\circ 20'$, | $b = 524$, | $c = 802$. |
| 6. $C = 64^\circ 50'$, | $a = 1\cdot72 \times 10^3$, | $c = 1\cdot75 \times 10^3$. |
| 7. $A = 20^\circ 52'$, | $a = 51\cdot3$, | $b = 142$. |
| 8. $A = 44^\circ 58'$, | $b = 41\cdot2$, | $c = 34\cdot7$. |
| 9. $C = 28^\circ 49'$. | $a = 153$, | $b = 83\cdot6$. |
| 10. $B = 113^\circ 11'$, | $a = 52\cdot0$, | $c = 111$. |

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|--------------------------|----------------------|-----------------------|
| 1. $A = 19^\circ 54'$, | $B = 40^\circ 24'$, | $C = 119^\circ 42'$. |
| 2. $A = 106^\circ 18'$, | $B = 42^\circ 24'$, | $C = 31^\circ 16'$. |
| 3. $A = 33^\circ 14'$, | $B = 96^\circ 28'$, | $C = 50^\circ 18'$. |

4. $A = 58^\circ 2'$, $B = 31^\circ 0'$, $C = 90^\circ 56'$.
5. $A = 110^\circ 14'$, $B = 30^\circ 6'$, $C = 39^\circ 40'$.
6. $A = 20^\circ 18'$, $B = 38^\circ 34'$, $C = 121^\circ 8'$.
7. $A = 32^\circ 14'$, $B = 94^\circ 40'$, $C = 53^\circ 6'$.
8. $A = 53^\circ 16'$, $B = 80^\circ 20'$, $C = 46^\circ 24'$.
9. $A = 38^\circ 38'$, $B = 96^\circ 26'$, $C = 44^\circ 58'$.
10. $A = 32^\circ 38'$, $B = 45^\circ 10'$, $C = 102^\circ 16'$.

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1. $c = 13.8$, $A = 90^\circ 24'$, $B = 36^\circ 12'$.
2. $a = 33.1$, $C = 62^\circ 22'$, $B = 54^\circ 0'$.
3. $c = 44.7$, $A = 39^\circ 9'$, $B = 28^\circ 27'$.
4. $b = 18.3$, $C = 65^\circ 37'$, $A = 46^\circ 41'$.
5. $c = 9.06 \times 10^3$, $B = 39^\circ 15'$, $A = 31^\circ 17'$.
6. $a = 365$, $C = 43^\circ 53'$, $B = 56^\circ 31'$.
7. $b = 13.4$, $C = 91^\circ 3'$, $A = 36^\circ 49'$.
8. $a = 3.02 \times 10^3$, $B = 98^\circ 42'$, $C = 55^\circ 54'$.
9. $b = 1.38 \times 10^3$, $C = 43^\circ 46'$, $A = 29^\circ 26'$.
10. $c = 16.8$, $A = 103^\circ 47'$, $B = 41^\circ 53'$.

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1. One. 2. Two. 3. None. 4. One.
5. $A_1 = 62^\circ 42'$ $C_1 = 62^\circ 42'$.
 $A_2 = 117^\circ 18'$, $C_2 = 8^\circ 6'$.
6. $A = 66^\circ 4'$, $B = 52^\circ 37'$.
7. $C_1 = 42^\circ 53'$ $B_1 = 97^\circ 22'$.
 $C_2 = 137^\circ 7'$, $B_2 = 3^\circ 8'$.
8. $A = 19^\circ 36'$, $B = 28^\circ 44'$, $a = 10.6$.
9. $B_1 = 49^\circ 38'$ $C_1 = 95^\circ 37'$ $c_1 = 23.8$.
 $B_2 = 130^\circ 22'$, $C_2 = 14^\circ 53'$, $c_2 = 6.15$.
10. $A_1 = 54^\circ 51'$ $C_1 = 83^\circ 34'$ $c_1 = 208$.
 $A_2 = 125^\circ 9'$, $C_2 = 13^\circ 16'$, $c_2 = 48.1$.

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1. $C = 101^\circ 32'$, $\Delta = 1.88 \times 10^3$.
2. $C = 36^\circ 32'$, $R = 7.04$.
3. $\Delta = 64.5$, altitude $= 11.9$.
4. $b = 4.82$, $R = 3.94$.
5. $34^\circ 46'$.
6. $R = 2.42$, $b = 4.30$.
7. $\Delta = 7.40 \times 10^3$, $R = 76.4$.
8. $B = 35^\circ 30'$, $R = 19.8$.
9. $a = 12.4$, $b = 10.8$, $c = 12.3$.
10. $R = 5.65$.
11. $CD = 4.05$, $\Delta DBC = 8.05$.
12. $OB = 3.24$ in., $R = 2.76$ in.

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13. $a = 4.50$ in., $b = 3.72$ in., $c = 4.64$ in.
14. $x = 7.00$ in., $y = 9.43$ in., $z = 10.7$ in.
15. $AX = 34.4$, $AY = 78.8$.
17. 32.8 chains; 47.2 sq. chains.
18. 35.2 chains.
19. $\hat{X} = 39^\circ 24'$, $\hat{Y} = 81^\circ 1'$, area = 310 sq. yd.
20. 12.8 miles; $\hat{RPQ} = 53^\circ 40'$.
21. 13.4 yd.
22. 24.3 miles per hr.
23. $AD = 31.6$ chains, $BC = 27.0$ chains.
24. $QR = 12.0$ chains.
25. 21.8 ft.
26. $AC = 1.60 \times 10^3$ yd.; $BC = 1.04 \times 10^3$ yd.
27. 639 yd.

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1. 5.54 miles.
2. 469 ft.
3. 3.22×10^3 ft.
4. 1.41 miles.
5. $42^\circ 42'$.
6. $8^\circ 8'$.
7. 222 ft.
8. 594 ft.
9. 527 ft.
10. 57.1 nautical miles.
11. 21.9 ft.
16. $22^\circ 51'$ or $157^\circ 9'$.
17. $11^\circ 57'$.
33. $2a$.

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1. $225^\circ, 315^\circ$.
2. $135^\circ, 315^\circ$.
3. $60^\circ, 300^\circ$.
4. $30^\circ, 210^\circ$.
5. $60^\circ, 120^\circ, 240^\circ, 300^\circ$.
6. $45^\circ, 135^\circ, 225^\circ, 315^\circ$.
7. $30^\circ, 150^\circ, 210^\circ, 330^\circ$.
8. $60^\circ, 120^\circ$.
9. $30^\circ, 90^\circ, 150^\circ, 270^\circ$.
10. $45^\circ, 135^\circ, 180^\circ$.
11. $45^\circ, 180^\circ, 225^\circ$.
12. $30^\circ, 90^\circ, 210^\circ, 270^\circ$.
13. $30^\circ, 150^\circ$.
14. $30^\circ, 150^\circ, 270^\circ$.
15. $48^\circ 11', 138^\circ 36', 221^\circ 24', 311^\circ 49'$.
16. $56^\circ 19', 144^\circ 28', 236^\circ 19', 324^\circ 28'$.
17. $30^\circ, 60^\circ, 210^\circ, 240^\circ$.
18. $66^\circ 25', 75^\circ 31', 284^\circ 29', 293^\circ 35'$.
19. $41^\circ 24', 120^\circ, 240^\circ, 318^\circ 36'$.
20. $135^\circ, 141^\circ 20', 315^\circ, 321^\circ 20'$.
21. $210^\circ, 221^\circ 49', 318^\circ 11', 330^\circ$.
22. $36^\circ 52', 120^\circ, 240^\circ, 323^\circ 8'$.
23. $63^\circ 26', 161^\circ 34', 243^\circ 26', 341^\circ 34'$.
24. $19^\circ 28', 120^\circ, 160^\circ 32', 240^\circ$.
25. $30^\circ, 45^\circ, 135^\circ, 150^\circ, 210^\circ, 225^\circ, 315^\circ, 330^\circ$.
26. $61^\circ 34', 169^\circ 46', 241^\circ 34', 349^\circ 46'$.
27. $7^\circ 45', 172^\circ 15'$.
28. $139^\circ 58', 220^\circ 2'$.
29. $18^\circ 26', 45^\circ, 198^\circ 26', 225^\circ$.

30. $35^\circ 9', 120^\circ 24', 215^\circ 9', 300^\circ 24'.$
 31. $23^\circ 35', 156^\circ 25', 270^\circ.$
 32. $48^\circ 11', 311^\circ 49'.$
 33. $60^\circ, 300^\circ.$
 34. $48^\circ 35', 131^\circ 25', 221^\circ 49', 318^\circ 11'.$
 35. $30^\circ, 120^\circ, 300^\circ, 330^\circ.$
 36. $120^\circ, 225^\circ, 300^\circ, 315^\circ.$
 37. $78^\circ 28', 228^\circ 35', 281^\circ 32', 311^\circ 25'.$
 38. $45^\circ, 120^\circ, 300^\circ, 315^\circ.$
 39. $2n\pi + \frac{7\pi}{6}, 2n\pi - \frac{\pi}{6}.$
 40. $n\pi \pm \frac{\pi}{6}.$
 41. $2n\pi \pm \frac{\pi}{2}, 2n\pi \pm \frac{\pi}{3}.$
 42. $n\pi, 2n\pi \pm \frac{\pi}{6}.$
 43. $n\pi \pm \frac{\pi}{4}.$
 44. $2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}, 2n\pi \pm \frac{\pi}{3}.$
 45. $n\pi + \frac{\pi}{4}.$
 46. $n\frac{\pi}{3}.$
 47. $(2n+1)\frac{\pi}{6}.$
 48. $n\pi + \frac{\pi}{4}, n \cdot 180^\circ + 63^\circ 26'.$
 49. $n \cdot 360^\circ + 48^\circ 35', (2n+1) 180^\circ - 48^\circ 35', 2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}.$
 50. $2n\pi \pm \frac{\pi}{3}, (2n+1)\pi.$
 51. $n\pi - \frac{\pi}{3}, n\pi + \frac{\pi}{4}.$
 52. $n\pi - \frac{\pi}{4}, n \cdot 180^\circ - 18^\circ 26'.$

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1. $75^\circ, 105^\circ, 255^\circ, 285^\circ.$
 2. $75^\circ, 165^\circ, 255^\circ, 345^\circ.$
 3. $60^\circ, 300^\circ.$
 4. $150^\circ, 210^\circ.$
 5. $70^\circ 32', 180^\circ, 289^\circ 28'.$
 6. $60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ.$
 7. $30^\circ, 90^\circ, 180^\circ, 270^\circ, 330^\circ.$
 8. $45^\circ, 120^\circ, 135^\circ, 225^\circ, 240^\circ, 315^\circ.$
 9. $45^\circ, 90^\circ, 180^\circ, 225^\circ, 270^\circ.$
 10. $97^\circ 10', 262^\circ 50'.$
 11. $164^\circ 12'.$
 12. $30^\circ, 45^\circ, 135^\circ, 150^\circ, 225^\circ, 270^\circ, 315^\circ.$
 13. $15^\circ, 75^\circ, 90^\circ, 195^\circ, 255^\circ, 270^\circ.$
 14. $54^\circ 24', 135^\circ, 215^\circ 36', 324^\circ 24'.$
 15. $146^\circ 26', 213^\circ 34'.$
 16. $15^\circ 29', 22^\circ 30', 105^\circ 29', 112^\circ 30', 195^\circ 29', 202^\circ 30', 285^\circ 29', 292^\circ 30'.$
 17. $24^\circ 28', 45^\circ, 135^\circ, 155^\circ 32', 225^\circ, 315^\circ.$
 18. $14^\circ 29', 60^\circ, 90^\circ, 120^\circ, 165^\circ 31', 180^\circ, 240^\circ, 270^\circ, 300^\circ.$
 19. $20^\circ, 40^\circ, 140^\circ, 160^\circ.$
 20. $0^\circ, 90^\circ, 180^\circ.$
 21. $35^\circ 47', 125^\circ 47'.$
 22. $0^\circ, 10^\circ, 50^\circ, 130^\circ, 170^\circ, 180^\circ.$
 23. $0^\circ, 45^\circ, 120^\circ.$

24. $9^\circ, 27^\circ, 45^\circ, 81^\circ, 99^\circ, 135^\circ, 153^\circ, 171^\circ$.
 25. $0^\circ, 30^\circ, 60^\circ, 120^\circ, 150^\circ, 180^\circ$.
 26. $30^\circ, 45^\circ, 90^\circ, 135^\circ, 150^\circ$.
 27. $0^\circ, 20^\circ, 90^\circ, 100^\circ, 140^\circ, 180^\circ$.
 28. $13^\circ 17', 82^\circ 59', 103^\circ 17', 172^\circ 59'$.
 29. $22^\circ 30', 30^\circ, 67^\circ 30', 112^\circ 30', 150^\circ, 157^\circ 30'$.
 30. $26^\circ 34', 30^\circ, 90^\circ, 150^\circ$.
 31. $n\pi \pm \frac{\alpha}{2}$.
 32. $(2n+1)\frac{\pi}{3} \pm \frac{\beta}{3}$.
 33. $\frac{2n\pi}{5} + \frac{\pi}{10}, 2n\pi + \frac{\pi}{2}$.
 34. $n\frac{\pi}{2}$.
 35. $2n\pi - \frac{\pi}{6}, 2n\pi + \frac{7\pi}{6}, n \cdot 360^\circ + 36^\circ 52', n \cdot 360^\circ + 143^\circ 8'$.
 36. $n\pi \pm \frac{\pi}{4}, n\pi + \frac{\pi}{3}$.
 37. $(2n+1)\frac{\pi}{2}, (2n+1)\frac{\pi}{3}$.
 38. $n\frac{\pi}{2}, n\frac{\pi}{3}$.
 39. $n\pi, \frac{2n}{m}\pi \pm \frac{\pi}{6m}$.
 40. $\frac{n\pi}{q}, \frac{(4n+1)\pi}{2(p \pm q)}$.

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1. $53^\circ 8'$.
 2. $67^\circ 23'$.
 3. $60^\circ 27', 193^\circ 17'$.
 4. $66^\circ 26', 158^\circ 48'$.
 5. $81^\circ 25', 338^\circ 5'$.
 6. $112^\circ 38', 323^\circ 8'$.
 7. $103^\circ 29', 156^\circ 7'$.
 8. $15^\circ, 135^\circ, 255^\circ$.
 9. $11^\circ 14', 131^\circ 14', 251^\circ 14'$.
 10. $24^\circ 12', 102^\circ 40', 204^\circ 12', 282^\circ 40'$.
 11. $36^\circ 52', 241^\circ 56'$.
 12. $12^\circ 40', 90^\circ$.
 13. $90^\circ, 208^\circ 4'$.
 14. $61^\circ 56', 323^\circ 8'$.
 15. $59^\circ 2', 171^\circ 52', 239^\circ 2', 351^\circ 52'$.
 16. $40^\circ 10', 77^\circ 43', 160^\circ 10', 197^\circ 43', 280^\circ 10', 317^\circ 43'$.
 17. $52^\circ 2', 210^\circ 20'$.
 18. $33^\circ 28', 259^\circ 12'$.
 19. $77^\circ 20', 180^\circ$.
 20. $71^\circ 4', 180^\circ$.
 21. $2n\pi + \frac{\pi}{6}, 2n\pi - \frac{\pi}{2}$.
 22. $n\pi + \frac{\pi}{3}$.
 23. $n\frac{\pi}{2} + \frac{\pi}{8}, n\frac{\pi}{2}$.
 24. $2n\pi + \frac{5\pi}{12}, 2n\pi + \frac{\pi}{12}$.
 25. $2n\frac{\pi}{3} + \frac{5\pi}{36}, 2n\frac{\pi}{3} - \frac{11\pi}{36}$.
 26. $n \cdot 360^\circ + 246^\circ 56', n \cdot 360^\circ - 66^\circ 56'$.
 27. $\frac{2n}{m}\pi, \frac{2n}{m}\pi - \frac{\pi}{3m}$.
 28. $n\pi + \frac{\pi}{6}, n\pi - \frac{\pi}{12}$.
 29. $2n\pi + \frac{7\pi}{12}$.
 30. $2n\pi + \frac{\pi}{4} - \beta \pm \alpha$.

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